

INTRODUCTION TO QCD SUM RULE APPROACH

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In these lectures, I describe the techniques used within the QCD sum rule approach. The basic concepts of the approach are introduced using a simple model of quantum-mechanical oscillator in 2+1 dimensions. Then I discuss their field-theoretical extension and the construction of the operator product expansion for current correlators in QCD. The calculation of static parameters is illustrated on the example of sum rules in the vector and axial meson channels. Finally, the QCD sum rule calculation of the pion electromagnetic form factor is presented as an example of application of the method to dynamic characteristics of the hadrons.

1 Introduction

1.1 Introductory remarks

QCD sum rules, invented more than 20 years ago by Shifman, Vainshtein and Zakharov¹ are still a rather popular approach for theoretical study of the hadronic structure. Applied originally to the simplest static hadronic characteristics, like masses, leptonic widths etc., they were also used to calculate much more complicated things like hadronic wave functions and form factors.

To begin with, I will briefly discuss the basic questions

- Why we need QCD sum rules?
- What are QCD sum rules?
- How the QCD sum rules are working?

using for illustration the QCD calculation of the pion form factor. In subsequent sections, I will discuss in more detail the machinery of QCD sum rules in several cases: calculation of the properties of the lowest resonances in ρ -meson and π/A_1 channels, and analysis of the pion form factor in various kinematical situations.

1.2 Pion form factor in QCD

According to the factorization theorem^{2,3}, for sufficiently large momentum transfers, one can represent the pion form factor as a sum of terms of increasing complexity (see Fig.1). The first term (purely *soft* contribution, Fig.1a contains no short-distance (SD) subprocesses. The second term Fig.1b contains a *hard* gluon exchange. There are also corrections to the hard term: higher or-

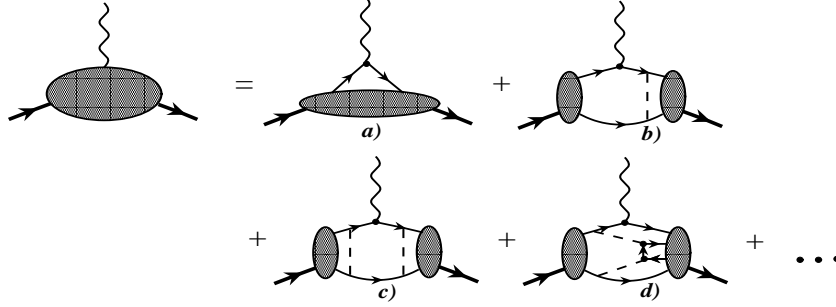


Figure 1: Structure of factorization for the pion form factor in QCD

der corrections Fig.1c containing extra α_s factors and higher twist corrections Fig.1d. In perturbative QCD one can calculate only the hard terms of this expansion.

The first result of perturbative QCD for exclusive processes is the prediction for the asymptotic behavior of the pion electromagnetic form factor ^{4,2,5,6}.

$$F_\pi(Q^2) = \int_0^1 dx \int_0^1 t(xP, (1-x)P; yP', (1-y)P', q) \varphi(x) \varphi(y) dy, \quad (1)$$

where $\varphi(x)$ is the pion wave function giving the probability amplitude to find the pion in a state where quarks carry fractions xP and $(1-x)P$ of its longitudinal momentum P ; $q = P' - P$ is the momentum transfer to the pion and $T(\dots)$ is the perturbatively calculable short-distance amplitude. The wave function $\varphi(x)$ accumulates long-distance information about the pion structure and cannot be calculated within the pQCD approach. Normally, some phenomenological assumptions are used about the form of this function depending on a single parameter x . Still, there is a challenge to calculate it from first principles of QCD.

To calculate the soft contribution to the form factor one should know something like the light-cone wave function $\psi(x, k_\perp)$ depending also on the transverse momentum k_\perp . I say “something like”, because in principle it is impossible to prove that the soft term reduces in QCD to the convolution

$$F(q^2) \sim \int \psi_P(x, k_\perp) \psi_P^*(x, k_\perp + xq) d^2 k_\perp dx. \quad (2)$$

In fact, one should treat the soft term as a nonfactorizable amplitude and the

problem is to calculate it in some reliable way.

The QCD sum rules are just giving the method to calculate the nonperturbative hadronic characteristics incorporating the asymptotic freedom property of QCD in a way similar to pQCD.

1.3 Basic ideas of the QCD sum rule approach

Among the existing approaches to the analysis of the nonperturbative effects in QCD the most close to perturbative QCD is the QCD sum rule method¹. Let us formulate its basic ideas within the context of the pion form factor problem.

It is evident that one cannot directly study the soft contribution with the on-shell pions, because then only long distances are involved. But perturbative QCD can be applied in a situation when all relevant momenta q , p_1 , p_2 are spacelike and sufficiently large: $|q^2|$, $|p_1^2|$, $|p_2^2| > 1 \text{ GeV}^2$. To describe the virtual pions one should use some interpolating field, the usual (for the QCD sum rule practitioners) choice being the axial current $j_5^\alpha = \bar{d}\gamma_5\gamma^\alpha u$. Its projection onto the pion state $|P, \pi\rangle$ is proportional to f_π :

$$\langle 0 | j_5^\alpha | P, \pi \rangle = i f_\pi P^\alpha. \quad (3)$$

Via the dispersion relation

$$T(p_1^2, p_2^2, q^2) = \frac{1}{\pi^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\rho(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)} \quad (4)$$

one can relate the amplitude $T(p_1^2, p_2^2, q^2)$ to its time-like counterpart $\rho(s_1, s_2, q^2)$ containing the double pole term

$$\rho_\pi(s_1, s_2, q^2) = \pi^2 f_\pi^2 \delta(s_1 - m_\pi^2) \delta(s_2 - m_\pi^2) F_\pi(Q^2) \quad (5)$$

corresponding to the pion form factor. However, the axial current has nonzero projections onto other hadronic states (A_1 -meson, say) as well, and the spectral density $\rho(s_1, s_2, q^2)$ contains also the part $\rho^{higher\ states}(s_1, s_2, q^2)$ related to other elastic and transition form factors. This is the price for going off the pion mass shell. The problem now is to pick out the F_π term from the whole mess.

Of course, calculating $T(p_1^2, p_2^2, q^2)$ in the lowest orders of perturbation theory one never observes something like the pion pole: one obtains a smooth function $\rho^{pert}(s_1, s_2, q^2)$ corresponding to transitions between the free-quark $\bar{u}d$ -states with invariant masses s_1 and s_2 , respectively. The difference between “exact” density $\rho(s_1, s_2, q^2)$ and its perturbative analog $\rho^{pert}(s_1, s_2, q^2)$

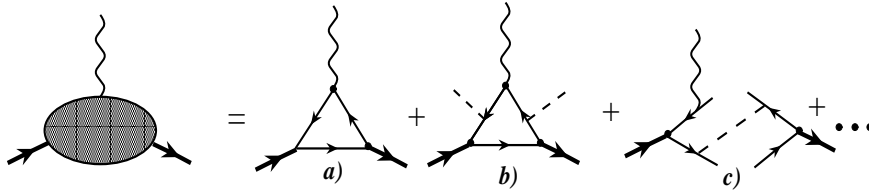


Figure 2: Structure of the operator product expansion for $T(p_1^2, p_2^2, q^2)$

is reflected by additional nonperturbative contributions to $T(p_1^2, p_2^2, q^2)$. These contributions are due to quark and gluon condensates $\langle \bar{q}q \rangle$, $\langle GG \rangle$ etc., describing (and/or parameterizing) the nontrivial structure of the QCD vacuum state. Formally, these terms appear from the operator product expansion for the amplitude $T(p_1^2, p_2^2, q^2)$ (Fig.2):

$$T(p_1^2, p_2^2, q^2) = T^{pert}(p_1^2, p_2^2, q^2) + a \frac{\langle GG \rangle}{(p^2)^3} + b \frac{\alpha_s \langle \bar{q}q \rangle^2}{(p^2)^4} + \dots \quad (6)$$

The problem now is to construct such a model of the spectral density $\rho(s_1, s_2, q^2)$ which gives the best agreement between two expressions for T , i.e., Eqs. (4) and (6). Naturally, having only a few first terms of the $1/p^2$ -expansion one can hope to reproduce only the gross features of the hadronic spectrum in the relevant channel. Still, just using the simple fact that the condensate contributions die out for large p^2 , one obtains the *global duality* relation between quark and hadronic densities

$$\int_0^\infty ds_1 \int_0^\infty ds_2 (\rho(s_1, s_2, q^2) - \rho^{pert}(s_1, s_2, q^2)) = 0. \quad (7)$$

In other words, integrally the two densities are rather similar. One can even expect that they are very close for high s to secure the convergence of the integral (7).

The standard ansatz is to approximate the higher states contribution into $\rho(s_1, s_2, q^2)$ by the free quark density (compare to the quasiclassical approximation for high levels in quantum mechanics):

$$\rho^{higher\ states}(s_1, s_2, q^2) = [1 - \theta(s_1 < s_0)\theta(s_2 < s_0)] \rho^{pert}(s_1, s_2, q^2), \quad (8)$$

with s_0 being the effective threshold for the higher states production.

1.4 Summary

To summarize, the basic ingredients of the QCD sum rule approach are

- Operator product expansion, or factorization of the short-distance, perturbatively calculable amplitudes, with the long-distance information accumulated by the quark and gluon condensates – the basic parameters of the approach – characterizing the properties of the QCD vacuum.
- Dispersion relations, relating the perturbatively calculated amplitude with the physical hadronic parameters.
- To extract the hadronic parameters one should construct an ansatz for the hadronic spectrum (like “first resonance plus continuum”, etc.) and then perform fitting procedure.

Just from the above list it is clear that QCD sum rules cannot produce exact results. Their precision is unavoidably limited by the following facts :

- One can calculate normally only few first terms of the OPE.
- The model ansatz reproduces the real density only approximately.
- Furthermore, in some channels could appear nonperturbative short-distance contributions (“direct instantons”).

As a result, using the QCD sum rules, one can normally expect only 10% - 20% accuracy.

2 QCD Sum Rules in Quantum Mechanics

We are going to apply the QCD sum rule method in the theory, where no exact results for hadrons were obtained so far. But, as formulated above, the QCD sum rule method is quite general. In fact, many features of the method can be demonstrated on the exactly solvable model of the quantum-mechanical oscillator. In the discussion below, we basically follow the pioneering paper⁷, with some simplifying modifications^{8,9}.

2.1 Sum rules for the lowest state of two-dimensional oscillator.

Let us consider the simplest example of a system with confinement, the oscillator, with the confining potential being

$$V(\vec{x}) = \frac{m\omega^2}{2}r^2. \quad (9)$$

Formulas are the simplest (without square roots, complicated factorial terms, etc.) if we choose the oscillator in two spatial dimensions (+ time dimension). In this case the energy levels are given by

$$E_n = (2n + 1)\omega, \quad (10)$$

and the values of the wave functions at the origin (quantum-mechanical analogues of f_π and other leptonic decay parameters) are the same for all levels corresponding to radial excitations with zero angular momentum:

$$|\psi_n(0)|^2 = \frac{m\omega}{\pi}. \quad (11)$$

To calculate E_0 and $\psi_0(0)$ using the sum rule approach one should proceed in a rather peculiar way: instead of concentrating on the lowest level, one should consider the weighted sum of all levels:

$$M(\epsilon) = \sum_{k=0}^{\infty} |\psi_k(0)|^2 e^{-E_k/\epsilon} \quad (12)$$

in which the lowest level dominates only for very small values of ϵ . For large ϵ all levels are essential. This combination (12) is directly related to the two-time Green function

$$G(x_1, t_1 | x_2, t_2) = \sum_{k=0}^{\infty} \psi_k^*(x_2) \psi_k(x_1) e^{-iE_k(t_2 - t_1)} \quad (13)$$

describing the probability amplitude for transition from the point x_1 at time t_1 to the point x_2 at time t_2 . To get $M(\epsilon)$ one should take $x_2 = x_1 = 0$, $t_1 = 0$ and $t_2 = 1/i\epsilon$. In other words, the transition is taken in the imaginary time.

In our case, explicit form of $M(\epsilon)$ can be easily calculated using Eqs. (10) and (11):

$$M^{osc}(\epsilon) = \sum_{k=0}^{\infty} \left(\frac{m\omega}{\pi} \right) e^{-(2k+1)\omega/\epsilon} = \frac{m\omega}{\pi \sinh(\omega/\epsilon)} \quad (14)$$

In the large- ϵ limit one has

$$M^{osc}(\epsilon) = \frac{m\epsilon}{2\pi} \left(1 - \frac{\omega^2}{6\epsilon^2} + \frac{7}{360} \frac{\omega^4}{\epsilon^4} - \frac{31}{15120} \frac{\omega^6}{\epsilon^6} + \dots \right) \quad (15)$$

i.e., the first term of the large- ϵ (\equiv small-time) expansion has no dependence on ω , the parameter characterizing the strength of oscillator potential. This simply means that at small time differences the Green function coincides with the free one:

$$G^{free}(2|1) = \frac{m}{2\pi i(t_2 - t_1)} \exp \left\{ -i \frac{m(x_2 - x_1)^2}{2(t_2 - t_1)} \right\} \quad (16)$$

and

$$M^{free}(\epsilon) = \frac{m\epsilon}{2\pi} \quad (17)$$

The corrections to the free-particle behavior in Eq. (15) are in powers of ω^2 . This corresponds to the perturbation expansion in the potential $V(r)$ (9):

$$M = M^{free} + M^{free} \otimes V \otimes M^{free} + M^{free} \otimes V \otimes M^{free} \otimes V \otimes M^{free} + \dots \quad (18)$$

The function $M^{free}(\epsilon)$ can also be represented as a sum over the (free) states with $E_k = k^2/2m$:

$$M^{free}(\epsilon) = \int \frac{d^2k}{(2\pi)^2} e^{-E_k/\epsilon} = \frac{m}{2\pi} \int_0^\infty e^{-E/\epsilon} dE \equiv \frac{1}{\pi} \int_0^\infty e^{-E/\epsilon} \rho(E) dE . \quad (19)$$

We introduced here a new function

$$\rho^{free}(E) = \frac{m}{2} \theta(E > 0), \quad (20)$$

the spectral density corresponding to the free motion, which will play an important role in the subsequent discussion. The oscillator function $M^{osc}(\epsilon)$ also can be written in the spectral representation with the spectral density $\rho^{osc}(E)$ which is in this case a superposition of the delta-functions:

$$\rho^{osc}(E) = m\omega \sum_{k=0}^{\infty} \delta(E - (2k+1)\omega). \quad (21)$$

At first sight, the two densities are completely different. However, if one integrates them from zero to the midpoint between the first and second levels, one obtains the same result:

$$\frac{1}{m} \int_0^{2\omega} \rho^{free}(E) dE = \omega = \frac{1}{m} \int_0^{2\omega} \rho^{osc}(E) dE . \quad (22)$$

Moreover, similar relations hold for other levels:

$$\int_{2k\omega}^{2(k+1)\omega} \rho^{free}(E) dE = m\omega = \int_{2k\omega}^{2(k+1)\omega} \rho^{osc}(E) dE . \quad (23)$$

The equality is not violated even if one multiplies $\rho(E)$ by an extra power of the energy E :

$$\int_{2k\omega}^{2(k+1)\omega} \rho^{free}(E) E dE = m\omega^2(2k+1) = \int_{2k\omega}^{2(k+1)\omega} \rho^{osc}(E) E dE . \quad (24)$$

Thus, there exists the *local duality* between each resonance and the free states. It looks like each level “takes” its part of the free spectral density, leaving the integral unchanged.

Turning back to the ω -expansion for $M^{osc}(\epsilon)$ (Eq. (15))

$$M^{osc}(\epsilon) = M^{free}(\epsilon) \left\{ 1 - \frac{\omega^2}{6\epsilon^2} + \frac{7}{360} \frac{\omega^4}{\epsilon^4} - \frac{31}{15120} \frac{\omega^6}{\epsilon^6} + \dots \right\} \quad (25)$$

one can write

$$M^{osc}(\epsilon) - M^{free}(\epsilon) = O(\omega^2/\epsilon) + O(\omega^4/\epsilon^3) + \dots \quad (26)$$

or, in terms of the spectral densities

$$\frac{1}{\pi} \int_0^\infty [\rho^{osc}(E) - \rho^{free}(E)] e^{-E/\epsilon} dE = \sum_{k=1} \frac{A_k}{\epsilon^k}. \quad (27)$$

The terms on the rhs, suppressed for large ϵ by powers of $1/\epsilon$ are normally referred to as “power corrections”. Thus, the difference between the integrated spectral densities vanishes as $\epsilon \rightarrow \infty$:

$$\frac{1}{\pi} \int_0^\infty [\rho^{osc}(E) - \rho^{free}(E)] dE = 0, \quad (28)$$

since there is no $O(1)$ term in the rhs of Eq. (27). Eq. (28) is a standard *global duality* relation. In quantum mechanics one can obtain such a relation for any potential regular at the origin.

The lesson is that the two densities, despite the apparent absence of similarity, are rather close to each other in an integral sense. Now, the idea is to incorporate this similarity in the sum rule

$$|\psi_0(0)|^2 e^{-E_0/\epsilon} + \text{“higher states”} = \frac{m\epsilon}{2\pi} \left\{ 1 - \frac{\omega^2}{6\epsilon^2} + \frac{7}{360} \frac{\omega^4}{\epsilon^4} - \frac{31}{15120} \frac{\omega^6}{\epsilon^6} + \dots \right\} \quad (29)$$

by substituting the “higher states” contribution by that of the free states having energy above some threshold s_0 . The sum rule then reads

$$|\tilde{\psi}_0(0)|^2 e^{-E_0/\epsilon} \approx \frac{\epsilon}{2} \left(1 - e^{-s_0/\epsilon} \right) - \frac{\omega^2}{12\epsilon} + \frac{7}{720} \frac{\omega^4}{\epsilon^3} - \frac{31}{30240} \frac{\omega^6}{\epsilon^5} + \dots \quad (30)$$

where

$$|\tilde{\psi}_0(0)|^2 = \frac{\pi}{m} |\psi_0(0)|^2$$

The parameters to be extracted from the sum rule are E_0 , $\tilde{\psi}_0(0)$ and s_0 . The first step is to extract E_0 . To this end we differentiate Eq. (30) with respect to $(-1/\epsilon)$ to get

$$|\tilde{\psi}_0(0)|^2 E_0 e^{-E_0/\epsilon} \approx \frac{\epsilon^2}{2} - \frac{\epsilon^2}{2} \left(1 + \frac{s_0}{\epsilon} \right) e^{-s_0/\epsilon} + \frac{\omega^2}{12} - \frac{7}{240} \frac{\omega^4}{\epsilon^2} + \frac{31}{6048} \frac{\omega^6}{\epsilon^4} + \dots \quad (31)$$

The ratio of the two equations just gives the expression for E_0 :

$$E_0 \approx \frac{\frac{\epsilon^2}{2} - \frac{\epsilon^2}{2}(1 + \frac{s_0}{\epsilon})e^{-s_0/\epsilon} + \frac{\omega^2}{12} - \frac{7}{240}\frac{\omega^4}{\epsilon^2} + \frac{31}{6048}\frac{\omega^6}{\epsilon^4} + \dots}{\frac{\epsilon}{2}(1 - e^{-s_0/\epsilon}) - \frac{\omega^2}{12\epsilon} + \frac{7}{720}\frac{\omega^4}{\epsilon^3} - \frac{31}{30240}\frac{\omega^6}{\epsilon^5} + \dots} . \quad (32)$$

If we would take the whole series in ω/ϵ and the exact expression for the higher states contribution, then the result for E_0 would be the ϵ -independent exact value $E_0 = \omega$. However, under approximations we made, the E_0 -value extracted from Eq. (32) depends both on the auxiliary parameter ϵ and the threshold parameter s_0 . The standard strategy is to take the value of s_0 , for which E_0 has the minimal dependence on ϵ in the intermediate region $1 < \epsilon/\omega < 2$ where both the higher power corrections and the error due to the rough model of the higher states contribution are small ($\approx 5 - 10\%$). Truncating the series by terms of the second order in ω^2 , one obtains^a from such a fitting $s_0 = 1.6\omega$ and $E_0 = 0.9\omega$. Adding the third order terms improves the result: $s_0 = 1.75\omega$ and $E_0 = 0.95\omega$. In our case one can substitute into the sum rule the all-order result. The remaining ϵ -dependence is then entirely due to the use of an approximate model for the higher states. For sufficiently small values of ϵ all the curves give the exact value $E_0 = 2\omega$. The flattest curves are those with $s_0 \sim 2\omega$.

The second step is to extract the $\tilde{\psi}(0)$ value from the original sum rule (30)

$$|\tilde{\psi}_0(0)|^2 \approx e^{E_0/\epsilon} \left\{ \frac{\epsilon}{2}(1 - e^{-s_0/\epsilon}) - \frac{\omega^2}{12\epsilon} + \frac{7}{720}\frac{\omega^4}{\epsilon^3} - \frac{31}{30240}\frac{\omega^6}{\epsilon^5} + \dots \right\} . \quad (33)$$

The analysis goes in a similar way. If one takes the third-order value $E_0 = 0.95\omega$, one obtains $|\tilde{\psi}_0(0)|^2 \sim 0.9\omega$. In the $\epsilon \rightarrow \infty$ limit one obtains the local duality relation

$$\{|\tilde{\psi}_0(0)|^2\}^{LD} = \frac{s_0}{2} . \quad (34)$$

So, the “exact value” for the duality interval s_0 is 2ω .

Summarizing, using the sum rule method one can determine the parameters of the lowest resonance with $\approx 10\%$ accuracy. The main source of the error is the use of a rather rough model for the spectral density of the higher states. It should be noted that in QCD situation is better: normally only the first resonance is narrow, while higher states are broader and broader, so that approximating them by free quark functions does not lead to large errors.

^aPlotting E_0 for various values of s_0 is absolutely straightforward, and we encourage the readers perform the fitting as an exercise.

2.2 Sum rules for form factors of the oscillator states.

To calculate the form factors

$$F_{kl}(Q) = \int \psi_k^*(x) \psi_l(x) e^{iQx} d^2x \quad (35)$$

within the sum rule approach, one should construct a similar integral using the Green functions (13)

$$G(x, 1/i\epsilon | 0, 0) = \sum_{k=0}^{\infty} \psi_k^*(x_2) \psi_k(x_1) e^{-E_k/\epsilon}. \quad (36)$$

This gives the amplitude

$$\begin{aligned} M(\epsilon_1, \epsilon_2, Q^2) &= \int G(x, 1/i\epsilon_1 | 0, 0) G^*(x, 1/i\epsilon_2 | 0, 0) e^{iQx} d^2x \\ &= \sum_{k,l=0}^{\infty} \psi_l^*(0) \psi_k(0) \left[\int e^{iQx} \psi_k^*(x) \psi_l(x) d^2x \right] e^{-E_k/\epsilon_1 - E_l/\epsilon_2} \\ &= \sum_{k,l=0}^{\infty} \psi_l^*(0) \psi_k(0) F_{kl}(Q^2) e^{-E_k/\epsilon_1 - E_l/\epsilon_2} \end{aligned} \quad (37)$$

One can write this amplitude in the (double) spectral representation:

$$M(\epsilon_1, \epsilon_2, Q^2) = \frac{1}{\pi^2} \int_0^\infty \int_0^\infty e^{-s_1/\epsilon_1 - s_2/\epsilon_2} \rho(s_1, s_2, Q^2) ds_1 ds_2 \quad (38)$$

where the spectral function is the sum of double delta-functions:

$$\rho(s_1, s_2, Q^2) = \pi^2 \sum_{k,l=0}^{\infty} \psi_l^*(0) \psi_k(0) F_{kl}(Q) \delta(s_1 - E_k) \delta(s_2 - E_l). \quad (39)$$

In the free case one has

$$G^{free}(x, 1/i\epsilon | 0, 0) = \frac{m\epsilon}{2\pi} e^{-mx^2\epsilon/2} = \int \frac{d^2k}{(2\pi)^2} e^{ikx} e^{-k^2/2m\epsilon} \quad (40)$$

for the Green function and

$$\begin{aligned} M^{free}(\epsilon_1, \epsilon_2, Q^2) &= \int \frac{d^2k}{(2\pi)^2} e^{-k^2/2m\epsilon_1 - (k+Q)^2/2m\epsilon_2} \\ &= \left(\frac{m}{2\pi}\right) \frac{\epsilon_1\epsilon_2}{\epsilon_1 + \epsilon_2} \exp\left(-\frac{Q^2}{2m(\epsilon_1 + \epsilon_2)}\right) \end{aligned} \quad (41)$$

for the amplitude. The free spectral function $\rho^{free}(s_1, s_2, Q^2)$ can be easily extracted:

$$\begin{aligned}\rho^{free}(s_1, s_2, Q^2) &= \pi^2 \int \frac{d^2k}{(2\pi)^2} \delta\left(s_1 - \frac{k^2}{2m}\right) \delta\left(s_2 - \frac{(k+Q)^2}{2m}\right) \\ &= \frac{\theta(|\sqrt{2ms_1} + \sqrt{2ms_2}| \geq |Q| \geq |\sqrt{2ms_1} - \sqrt{2ms_2}|)}{\sqrt{4s_1s_2 - (s_1 + s_2 - Q^2/2m)^2}}. \quad (42)\end{aligned}$$

The restriction on s_1, s_2 is in fact the triangle relation

$$|k_1| + |k_2| \geq |Q| \geq ||k_1| - |k_2||$$

between the initial momentum k_1 ($|k_1| = \sqrt{2ms_1}$), the momentum transfer Q and the final momentum $k_2 = (k_1 + Q)$ ($|k_2| = \sqrt{2ms_2}$).

The oscillator Green function is also known:

$$G^{osc}(x, 1/i\epsilon | 0, 0) = \frac{m\omega}{2\pi \sinh(\omega/\epsilon)} \exp\left(-\frac{m\omega \vec{x}^2 \cosh(\omega/\epsilon)}{2 \sinh(\omega/\epsilon)}\right) \quad (43)$$

and using it, one can calculate the amplitude

$$M(\epsilon_1, \epsilon_2, Q^2) = \frac{m\omega}{2\pi \sinh(\omega/\epsilon_1 + \omega/\epsilon_2)} \exp\left(-\frac{Q^2}{2m\omega} \frac{\sinh(\omega/\epsilon_1) \sinh(\omega/\epsilon_2)}{\sinh(\omega/\epsilon_1 + \omega/\epsilon_2)}\right). \quad (44)$$

Now, representing M as

$$\begin{aligned}M(\epsilon_1, \epsilon_2, Q^2) &= \frac{m\omega}{\pi} \sum_{k=0}^{\infty} e^{-(2k+1)\omega(1/\epsilon_1 + 1/\epsilon_2)} \exp\left(-\frac{Q^2}{4m\omega}\right) \\ &\quad \exp\left(\frac{Q^2}{4m\omega} \left[1 - \frac{(1 - e^{-\omega/\epsilon_1})(1 - e^{-\omega/\epsilon_2})}{(1 - e^{-\omega/\epsilon_1}e^{-\omega/\epsilon_2})}\right]\right) \quad (45)\end{aligned}$$

and expanding M in powers of $e^{-\omega/\epsilon_1} \equiv \mathcal{E}_1$ and $e^{-\omega/\epsilon_2} \equiv \mathcal{E}_2$ one can extract the form factors $F_{lk}(Q^2)$:

$$\begin{aligned}M(\epsilon_1, \epsilon_2, Q^2) &= \frac{m\omega}{\pi} \sum_{k=0}^{\infty} \mathcal{E}_1^{2k+1} \mathcal{E}_2^{2k+1} \exp\left(-\frac{Q^2}{4m\omega}\right) \\ &\quad \left\{1 + \frac{Q^2}{4m\omega} \left[1 - \frac{(1 - \mathcal{E}_1)(1 - \mathcal{E}_2)}{1 - \mathcal{E}_1\mathcal{E}_2}\right] + \frac{1}{2} \left(\frac{Q^2}{4m\omega}\right)^2 [\dots]^2 + \dots\right\} \quad (46)\end{aligned}$$

identifying them as the factors in front of the $\mathcal{E}_1^k \mathcal{E}_2^l$ terms:

$$F_{00}(Q^2) = \exp\left(-\frac{Q^2}{4m\omega}\right),$$

$$\begin{aligned}
F_{01}(Q^2) &= F_{10}(Q^2) = \frac{Q^2}{4m\omega} \exp\left(-\frac{Q^2}{4m\omega}\right), \\
F_{11}(Q^2) &= \left(1 - \frac{Q^2}{4m\omega}\right)^2 \exp\left(-\frac{Q^2}{4m\omega}\right)
\end{aligned} \tag{47}$$

It is instructive to write down the amplitude $M(\epsilon_1, \epsilon_2, Q^2)$ for $Q^2 = 0$:

$$M(\epsilon_1, \epsilon_2, Q^2 = 0) = \frac{m\omega}{2\pi \sinh(\omega/\epsilon_1 + \omega/\epsilon_2)} = \frac{m\omega}{\pi} \sum_{k=0}^{\infty} \mathcal{E}_1^{2k+1} \mathcal{E}_2^{2k+1}. \tag{48}$$

It is easy to notice that

- There are no terms with $k \neq l$: only the diagonal transitions are allowed in this limit
- The numerical factor is just the same as in Eq. (14) for $M^{osc}(\epsilon)$. This means that $F_{kk}(0) = 1$ for all k , just as required by charge conservation.

Next step is to get the perturbative expansion in powers of ω :

$$\begin{aligned}
M(\epsilon_1, \epsilon_2, Q^2) &= \frac{m\epsilon_1\epsilon_2}{2\pi(\epsilon_1 + \epsilon_2)} \left\{ \left[1 - \frac{\omega^2}{6} \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \right)^2 + \dots \right] \right. \\
&\quad \times \exp \left(-\frac{Q^2}{2m} \frac{\frac{1}{\epsilon_1}(1 + \frac{1}{6}\frac{\omega^2}{\epsilon_1^2} + \dots)\frac{1}{\epsilon_2}(1 + \frac{1}{6}\frac{\omega^2}{\epsilon_2^2} + \dots)}{(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2})(1 + \frac{\omega^2}{6}(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2})^2 + \dots)} \right) \Big\} \\
&= \frac{m\epsilon_1\epsilon_2}{2\pi(\epsilon_1 + \epsilon_2)} \left\{ \left[1 - \frac{\omega^2}{6} \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \right)^2 + \frac{7\omega^4}{360} \left(\frac{1}{\epsilon_1} + \frac{1}{\epsilon_2} \right)^4 + \dots \right] \right. \\
&\quad \exp \left(-\frac{Q^2}{2m(\epsilon_1 + \epsilon_2)} \right) \left[1 + \frac{Q^2\omega^2}{6m\epsilon_1\epsilon_2(\epsilon_1 + \epsilon_2)} \right. \\
&\quad \left. \left. - \frac{Q^2\omega^4[(\epsilon_1 + \epsilon_2)^2 + 2\epsilon_1\epsilon_2]}{90m\epsilon_1^3\epsilon_2^3(\epsilon_1 + \epsilon_2)} + \frac{Q^4\omega^4}{72m^2\epsilon_1^2\epsilon_2^2(\epsilon_1 + \epsilon_2)^2} + \dots \right] \right\} \\
&= M^{free}(\epsilon_1, \epsilon_2, Q^2) \left\{ 1 + \sum_{k=1}^{\infty} (\omega^2)^k \varphi_k(\epsilon_1, \epsilon_2, Q^2) \right\} \tag{49}
\end{aligned}$$

The sum rule now reads

$$\begin{aligned}
|\psi_0(0)|^2 F_{00}(Q^2) e^{-E_0/\epsilon_1 - E_0/\epsilon_2} &+ \text{“higher states”} \\
&= M^{free}(\epsilon_1, \epsilon_2, Q^2) \left\{ 1 + O\left(\frac{\omega^2}{\epsilon^2}\right) + O\left(\frac{\omega^4}{\epsilon^4}\right) + \dots \right\} \tag{50}
\end{aligned}$$

To proceed further, one should take some ansatz for the higher states contribution. The ansatz similar to that used earlier is to assume that

$$“higher\ states” = “free\ states”$$

outside the square $(0, s_0) \otimes (0, s_0)$. Then the sum rule has the form

$$\begin{aligned} |\psi_0(0)|^2 F_{00}(Q^2) e^{-E_0/\epsilon_1 - E_0/\epsilon_2} \\ = \frac{1}{\pi^2} \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \rho^{free}(s_1, s_2, Q^2) e^{-s_1/\epsilon_1 - s_2/\epsilon_2} \\ + O\left(\frac{\omega^2}{\epsilon}\right) + O\left(\frac{\omega^4}{\epsilon^3}\right) + \dots \end{aligned} \quad (51)$$

In the $\epsilon_1, \epsilon_2 \rightarrow \infty$ limit one obtains the local duality relation:

$$|\psi_0(0)|^2 F_{00}(Q^2) = \frac{1}{\pi^2} \int_0^{s_0} \int_0^{s_0} \rho^{free}(s_1, s_2, Q^2) ds_1 ds_2 \quad (52)$$

with $\rho^{free}(s_1, s_2, Q^2)$ given by Eq. (42):

$$\rho^{free}(s_1, s_2, Q^2) = \frac{1}{4} \int d^2 k \delta\left(s_1 - \frac{k^2}{2m}\right) \delta\left(s_2 - \frac{(k+Q)^2}{2m}\right). \quad (53)$$

Using the two expressions given above, one can write

$$|\psi_0(0)|^2 F_{00}^{LD}(Q^2) = \frac{1}{(2\pi)^2} \int d^2 k \theta\left(\frac{k^2}{2m} < s_0\right) \theta\left(\frac{(k+Q)^2}{2m} < s_0\right). \quad (54)$$

This integral can be easily calculated if one notices that it defines the overlap area of two circles having the same radius $R^2 = 2ms_0$, with their centers separated by the distance equal to Q :

$$|\psi_0(0)|^2 F_{00}^{LD}(Q^2) = \frac{ms_0}{\pi^2} \left[\cos^{-1}(\kappa) - \kappa \sqrt{1 - \kappa^2} \right] \theta(Q^2 < 8ms_0), \quad (55)$$

where $\kappa = Q/\sqrt{8ms_0}$. Recall now that from the local duality relation for $M(\epsilon)$ we had

$$\{|\psi_0(0)|^2\}^{LD} = \frac{ms_0}{2\pi} \quad (56)$$

and this gives the result

$$F_{00}^{LD}(Q^2) = \frac{2}{\pi} \left[\cos^{-1}(\kappa) - \kappa \sqrt{1 - \kappa^2} \right] \theta(Q^2 < 8ms_0) \quad (57)$$

having the correct normalization $F_{00}^{LD}(Q^2) = 1$.

The local duality prediction calculated for the “exact value” $s_0 = 2\omega$ of the duality interval

$$F_{00}^{LD}(Q^2)|_{s_0=2\omega} = \frac{2}{\pi} \left[\cos^{-1} \left(\frac{Q}{\sqrt{16m\omega}} \right) - \frac{Q}{\sqrt{16m\omega}} \sqrt{1 - \frac{Q^2}{16m\omega}} \right] \theta(Q^2 < 16m\omega) \quad (58)$$

is very close to the exact form factor

$$F_{00}^{exact}(Q^2) = \exp \left(-\frac{Q^2}{4m\omega} \right). \quad (59)$$

There is a very simple physics behind the local duality. One should just compare the local duality prescription for the form factor

$$F_{00}^{LD}(Q^2) = \frac{1}{|\psi_0(0)|^2} \int \frac{d^2k}{(2\pi)^2} \theta \left(\frac{k^2}{2m} < s_0 \right) \theta \left(\frac{(k+Q)^2}{2m} < s_0 \right) \quad (60)$$

and the exact formula

$$F_{00}(Q^2) = \int \frac{d^2k}{(2\pi)^2} \psi_0(k) \psi_0^*(k+Q) \quad (61)$$

to realize that the local duality, in fact, is equivalent to using a model form for the wave function in the momentum space:

$$\tilde{\psi}_0^{LD}(k) = \frac{1}{|\psi_0(0)|} \theta(k^2 < 2ms_0) \quad (62)$$

or, taking $s_0 = 2\omega$:

$$\tilde{\psi}_0^{LD}(k) = \sqrt{\frac{\pi}{m\omega}} \theta(k^2 < 4m\omega), \quad (63)$$

to be compared with the exact wave function

$$\tilde{\psi}_0(k) = 2\sqrt{\frac{\pi}{m\omega}} \exp \left(-\frac{k^2}{2m\omega} \right). \quad (64)$$

It is easy to establish that the model wave function correctly reproduces the simplest integral properties of the exact wave function:

$$\int \tilde{\psi}^{exact}(k) d^2k = \int \tilde{\psi}^{LD}(k) d^2k \quad (65)$$

$$\int |\tilde{\psi}^{exact}(k)|^2 d^2k = \int |\tilde{\psi}^{LD}(k)|^2 d^2k. \quad (66)$$

3 Condensates and Operator Product Expansion

3.1 Correlators

The basic object in the quantum mechanical example was the Green function $G(x_1, t_1 | x_2, t_2)$ describing the propagation of a particle from one point to another. In fact, it was sufficient to analyze the probability amplitude for the particle to remain at the same spatial point after some imaginary time elapsed:

$$M(\epsilon) = G(0, 0 | 0, 1/i\epsilon) . \quad (67)$$

The question is what should be its analog in the quantum field theory? Recall, that the motion in the external potential corresponds to a two-particle system with one of the particles being infinitely massive. If this particle moves, the other moves together with it. So, the Green function describes really a collective motion of two particles from one point to another. In quantum field theory this corresponds to the Green-like functions

$$\langle 0 | T(j(x) j(y)) | 0 \rangle$$

normally referred to as correlators, with the currents $j(x)$ being the products of fields at the same point, *e.g.* $J^\mu(x) = \bar{q}(x)\gamma^\mu q(x)$. Due to the translation invariance

$$\langle 0 | T(j(x) j(y)) | 0 \rangle \rightarrow \langle 0 | T(j(x-y) j(0)) | 0 \rangle$$

it is always possible and convenient to take one point at the origin. In QFT, it is also more convenient to consider correlators in the momentum representation:

$$\Pi(q) = \int e^{iqx} \langle 0 | T(j(x) j(0)) | 0 \rangle d^4x .$$

In the lowest order the correlator is given by the product of the relevant propagators, but there are also radiative corrections, which one can calculate in perturbative QCD, at least in the region where q is spacelike $q^2 < 0$ and large enough.

Next step is to relate $\Pi(q)$ to a sum over the physical states. This is performed *via* the dispersion relation

$$\Pi(q^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho(s)}{s - q^2} ds + \text{“subtractions”} .$$

The subtraction terms appear if the spectral density is such that the dispersion integral diverges. If, say, $\rho(s) \sim s^n$, one needs $(n+1)$ subtractions:

$$\Pi(q^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho(s)}{s - q^2} ds$$

$$\begin{aligned}
& -\frac{1}{\pi} \int_0^\infty ds \rho(s) \left\{ \frac{1}{s-\lambda^2} + \frac{q^2-\lambda^2}{(s-\lambda^2)^2} + \dots + \frac{(q^2-\lambda^2)^n}{(s-\lambda^2)^{n+1}} \right\} \\
& + \left\{ \Pi(\lambda^2) + (q^2-\lambda^2)\Pi'(\lambda^2) + \dots + \frac{(q^2-\lambda^2)^n}{n!} \Pi^{(n)}(\lambda^2) \right\}. \quad (68)
\end{aligned}$$

After collecting all the terms containing $\rho(s)$, one observes that $\rho(s)$ is divided by $(s-q^2)(s-\lambda^2)^{n+1}$, and the integral converges. Note, that all the subtraction terms are polynomials of a finite order in Q^2 (from now on we use the notation $q^2 = -Q^2$, with $Q^2 > 0$).

3.2 Borel transformation

Note, that in the oscillator studies, we dealt with the exponentially weighted sum:

$$M(\epsilon) = \frac{1}{\pi} \int_0^\infty e^{-s/\epsilon} \rho^{osc}(s) ds. \quad (69)$$

We observed then that it has a nice property that for ϵ values of an order of ω , both the contributions of the higher states were strongly damped and the perturbative expansion for $M(\epsilon)$ was converging fast enough. With the power weight implied by the dispersion relation, the suppression of the higher states contribution is not so effective.

However, it is rather easy to transform the original dispersion integral into the exponentially weighted one. To this end, one should apply the so-called Borel transformation, formally defined by the formula¹

$$\Phi(M^2) = \lim_{\substack{Q^2, n \rightarrow \infty \\ Q^2/n = M^2}} \frac{1}{(n-1)!} (Q^2)^n \left[-\frac{d}{dQ^2} \right]^n \Pi(Q^2). \quad (70)$$

It is easy to check that applying the above procedure to $1/(s+Q^2)$ one really obtains an exponential. Indeed,

$$\left[-\frac{d}{dQ^2} \right]^n \frac{1}{s+Q^2} = \frac{n!}{(s+Q^2)^{n+1}}. \quad (71)$$

Multiplying the result by $\frac{(Q^2)^n}{(n-1)!}$ one obtains

$$n \frac{(Q^2)^n}{(s+Q^2)^{n+1}} = \frac{1}{Q^2/n} \frac{1}{(1+s/Q^2)^{n+1}} = \frac{1}{M^2} \frac{1}{(1+s/nM^2)^{n+1}}. \quad (72)$$

Finally, taking the $n \rightarrow \infty$ limit one obtains $\frac{1}{M^2} e^{-s/M^2}$. This means that

$$\Phi(M^2) \equiv B(Q^2 \rightarrow M^2) \Pi(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{ds}{M^2} \rho(s) e^{-s/M^2} + \text{“nothing”} \quad (73)$$

where “nothing” means that all the subtraction terms have disappeared, since no polynomial can survive the infinite number of differentiations. Thus, the function $\Phi(M^2)$ is the QFT analog of $M(\epsilon)$.

In many cases one can perform the Borel transformation using the formula¹⁰

$$B(Q^2 \rightarrow M^2)\{e^{-AQ^2}\} = \delta(1 - AM^2). \quad (74)$$

As an illustration, let us reproduce the result obtained above :

$$\begin{aligned} B \frac{1}{s + Q^2} &= B \int_0^\infty e^{-\alpha(s+Q^2)} d\alpha \\ &= \int_0^\infty e^{-\alpha s} \delta(1 - \alpha M^2) d\alpha = \frac{1}{M^2} e^{-s/M^2}. \end{aligned} \quad (75)$$

Other functions, the Borel transforms of which we will need, are those having a power behavior in $1/Q^2$. In this case we have

$$\begin{aligned} B \frac{1}{(Q^2)^n} &= B \int_0^\infty e^{-\alpha Q^2} \frac{\alpha^{n-1}}{(n-1)!} d\alpha \\ &= \int_0^\infty \delta(1 - \alpha M^2) \frac{\alpha^{n-1}}{(n-1)!} d\alpha = \frac{1}{(n-1)!(M^2)^n}. \end{aligned} \quad (76)$$

Thus, a power expansion for $\Phi(M^2)$ over $1/M^2$ converges much faster than the original expansion of $\Pi(Q^2)$ over $1/Q^2$, since each term is now suppressed factorially. This looks precisely like the Borel improvement of a series, and that is why Shifman, Vainshtein and Zakharov¹ referred to the transformation they introduced as “the Borel transformation”. The representation for $\Phi(M^2)$ has the form of the Laplace transformation, and that is why some people call the sum rules obtained in this way “the Laplace transform sum rules”. In fact, there is no contradiction: $\Phi(M^2)$ is the Borel transform of $\Pi(Q^2)$ and the Laplace transform of $\rho(s)$.

To summarize, the Borel transformation produces two improvements:

- higher state contributions in the spectral representation for $\Phi(M^2)$ are exponentially suppressed,
- its power series expansion in $1/M^2$ is factorially improved.

3.3 Heavy-light system and the $m \rightarrow \infty$ limit

Using the basic formula, it is rather easy to establish¹¹ that in the limit when one of the particles becomes infinitely massive, the QFT function $\Phi(M^2)$ converts into its quantum-mechanical analog. Consider the lowest-order diagram.

The propagator of the heavy particle is

$$\frac{1}{(q+k)^2 - m^2} = \frac{1}{q^2 - m^2 + 2qk + k^2} .$$

In the $m \rightarrow \infty$ limit all the bound state masses are equal to this mass plus some finite quantity. So, it makes sense to subtract this infinite constant. To this end one can represent the total momentum q as

$$q = mv + l$$

where v is the four-velocity. Then

$$q^2 - m^2 = 2m(vl) + l^2$$

The scalar product $(vl) \equiv E$ just has the meaning of the energy variable. There is only one more $O(m)$ term in the propagator:

$$2(qk) = 2m(vk)\{1 + O(1/m)\} .$$

With this accuracy, one can write down the relevant Feynman integral as

$$I(m, v, E) = \int \frac{d^4k}{2m(E + (vk))} S(k) ,$$

where $S(k)$ is the propagator of the light particle. Now, applying the Borel transformation $B(E \rightarrow \epsilon)$ which converts $1/(E + (vk))$ into $\frac{1}{\epsilon} e^{-(vk)/\epsilon}$ we get

$$M(m, v, \epsilon) \equiv B(E \rightarrow \epsilon)I(m, v, E) = \frac{1}{2m\epsilon} \int e^{-(vk)/\epsilon} S(k) d^4k .$$

It is easy to realize that the integral above is just the light particle propagator in the configuration space

$$M(m, v, \epsilon) = \frac{1}{2m\epsilon} \langle \varphi(0) \varphi(iv/\epsilon) \rangle$$

i.e., the Green function describing propagation in the imaginary time, since the velocity vector v^μ , in the rest frame of the heavy particle, is oriented exactly in the time direction.

Thus, we have demonstrated that the Borel transformed correlators are just the QFT generalization of the imaginary time Green functions we studied in the quantum-mechanical examples.

3.4 Perturbative vs nonperturbative contributions in QCD

From the technical point of view, the strategy now is

- to calculate the correlator $\Pi(Q^2)$ in the deep spacelike region where one can rely on the asymptotic freedom of QCD,
- apply the Borel transformation to the resulting expression to get a $1/M^2$ expansion for $\Phi(M^2)$,
- finally, using the dispersion representation for $\Phi(M^2)$, to construct a sum rule.

The question is, what are the contributions analogous to the ω^2/ϵ^2 corrections?

The standard idea about the QCD potential $V(r)$ is that it consists of a Coulomb-like part $\sim \alpha_s(1/r^2\Lambda^2)$ calculable perturbatively at short distances and a nonperturbative $\sim r$ long distance “confining” part:

$$V(r) = V^{\text{“Coulomb”}}(r) + V^{\text{conf}}(r)$$

The magnitude of the “Coulomb” effects is determined by the QCD running coupling constant

$$\alpha_s(Q^2) = \frac{4\pi}{9 \log(Q^2/\Lambda^2)} + \dots$$

The standard modern estimate for the value of the QCD scale Λ is

$$\Lambda \sim 200 \text{ MeV} .$$

As we will see, the parameters that really appear in QCD sum rules have dimension of $(\text{mass})^2$, and one should compare

$$\Lambda^2 \sim 0.04 \text{ GeV}^2$$

to the typical hadronic scales like ρ and nucleon masses (squared):

$$m_\rho^2 \approx 0.6 \text{ GeV}^2, \quad m_N^2 \approx 0.9 \text{ GeV}^2.$$

To get such scales from Λ^2 one needs unnaturally large numerical factors of an order of 25 or even 100. So, it is most unlikely that the $O(\alpha_s)$ -corrections can generate scales of an order of 1 GeV and, hence, the “Coulomb” part is practically irrelevant to the formation of the spectrum of light hadrons. Our problem now is to take into account the effects due to the long-range nonperturbative part of the QCD potential.

Our study of the quantum-mechanical oscillator (which is the simplest example of a system with confinement) has demonstrated that the propagation

amplitude in the presence of the potential is different from the free Green function. Furthermore, we observed that the difference vanishes at short distances and that one can calculate exact Green function perturbatively, expanding in powers of the oscillator potential. In QCD, with two components of $V(r)$ one can imagine a double series, both in $V^{\text{“Coulomb”}}(r)$ and $V^{\text{conf}}(r)$. There is no problem to arrange an expansion in $V^{\text{“Coulomb”}}(r)$, – this will be just the standard perturbation theory in the QCD coupling constant. On the other hand, the confining potential $V^{\text{conf}}(r)$ in QCD is not even known. Moreover, the widespread belief is that it is determined by some essentially nonperturbative effects, *i.e.*, by those which cannot be seen in perturbation theory.

In such a situation, a possible way out is to proceed as follows:

- to construct perturbation expansion in terms of quark and gluon propagators (this amounts to taking into account only the Coulombic part of the potential),
- to postulate that quark and gluon propagators are modified by the long-range confinement part of the QCD potential; but the modification is soft in a sense that at short distances the difference between exact and perturbative (free-field) propagators vanishes.

To formalize this statement, one can write the exact propagator $\mathcal{D}^{\text{exact}}(x)$ as a vacuum average of a T-product of fields in the exact vacuum Ω (corresponding to a confining potential)

$$\mathcal{D}^{\text{exact}}(x) = \langle \Omega | T(\varphi(x)\varphi(0)) | \Omega \rangle.$$

According to the Wick theorem, one can write the T -product as the sum

$$T(\varphi(x)\varphi(0)) = \underbrace{\varphi(x)\varphi(0)} + : \varphi(x)\varphi(0) :$$

of the “pairing” and the “normal” product. The “pairing” is just the expectation value of the T -product over the perturbative vacuum

$$\underbrace{\varphi(x)\varphi(0)} = \langle 0 | T(\varphi(x)\varphi(0)) | 0 \rangle.$$

i.e., the perturbative propagator. By this definition, the normal product $: \varphi(x)\varphi(0) :$ vanishes if averaged over the perturbative vacuum:

$$\langle 0 | : \varphi(x)\varphi(0) : | 0 \rangle = 0.$$

Thus, our assumption that $\mathcal{D}^{\text{exact}}(x) \neq \mathcal{D}^{\text{pert}}(x)$ is equivalent to the statement

$$\langle \Omega | : \varphi(x)\varphi(0) : | \Omega \rangle \neq 0,$$

which is the starting point to calculating power corrections in QCD. To simplify the notation, in what follows we will write the lhs of the above equation as $\langle \varphi(x)\varphi(0) \rangle$.

3.5 Condensates

In the oscillator case the analog of $\langle \varphi(x)\varphi(0) \rangle$ is the difference between the exact Green function

$$G^{osc}(0, 0|0, \tau/i) = \frac{m\omega}{2\pi \sinh(\omega\tau)}$$

and the free Green function

$$G^{free}(0, 0|0, \tau/i) = \frac{m}{2\pi\tau}$$

where $\tau \equiv 1/\epsilon$ is just the imaginary time variable. Note, that both oscillator and free Green function are singular for $\tau \rightarrow 0$, but the difference

$$\begin{aligned} & G^{osc}(0, 0|0, \tau/i) - G^{free}(0, 0|0, \tau/i) \\ &= \frac{m}{2\pi} \left[-\frac{1}{6}\omega^2\tau + \frac{7}{360}\omega^4\tau^3 - \frac{31}{15120}\omega^6\tau^5 + \dots \right] \end{aligned} \quad (77)$$

is regular at that point, and one can even expand the difference into the Taylor series in $(\omega\tau)^2$. The expansion is possible essentially because the confining potential is regular (and even vanishing) at $\tau = 0$.

In the QCD sum rule approach it is also assumed that the confinement effects are sufficiently soft to allow for the Taylor expansion of $\langle \varphi(x)\varphi(0) \rangle$ at $x = 0$:

$$\begin{aligned} \langle \varphi(0)\varphi(x) \rangle &= \sum_{n=0}^{\infty} \frac{x^{\mu_1} \dots x^{\mu_n}}{n!} \langle \varphi(0)\partial_{\mu_1} \dots \partial_{\mu_n}\varphi(0) \rangle \\ &= \langle \varphi\varphi \rangle + x^\mu \langle \varphi\partial_\mu\varphi \rangle + \frac{x^{\mu_1}x^{\mu_2}}{2} \langle \varphi\partial_{\mu_1}\partial_{\mu_2}\varphi \rangle + \dots \end{aligned} \quad (78)$$

This is, in fact, the expansion of the nonlocal object $\langle \varphi(0)\varphi(x) \rangle$ over the vacuum matrix elements of the local composite operators. Not all operators really contribute to the expansion above. Using the Lorentz covariance, it is trivial to find that

$$\begin{aligned} \langle \varphi\partial_\mu\varphi \rangle &= 0, \\ \langle \varphi\partial_{\mu_1}\partial_{\mu_2}\varphi \rangle &= \frac{1}{4}g_{\mu_1\mu_2} \langle \varphi\partial^2\varphi \rangle, \end{aligned} \quad (79)$$

etc., so that finally one arrives at the expansion in x^2 :

$$\langle \varphi(0)\varphi(x) \rangle = \sum_{n=0}^{\infty} \left(\frac{x^2}{4} \right)^n \frac{1}{n!(n+1)!} \langle \varphi(\partial^2)^n\varphi \rangle. \quad (80)$$

Thus, the modification of the propagator by the nonperturbative effects is now parametrized by the matrix elements of the composite operators like $\langle \varphi(\partial^2)^n \varphi \rangle$. The examples in QCD are

- $\langle \bar{q}q \rangle$ referred to as the quark condensate,
- $\langle \bar{q}D^2q \rangle$, characterizing the average virtuality of the vacuum quarks,
- the gluon condensate $\langle G_{\mu\nu}^a G_{\mu\nu}^a \rangle$, etc. Here $D_\mu \equiv \partial_\mu - igA_\mu$ is the covariant derivative and $G_{\mu\nu} = (i/g)[D_\mu, D_\nu]$ is the gluonic field strength. Note, that only gauge invariant composite operators should appear in QCD, *i.e.* each ∂_μ must be accompanied by the relevant A_μ . How this happens, will be discussed later.

3.6 Operator product expansion

It is instructive to analyze the above expansion in the momentum representation

$$\langle \varphi(0)\varphi(x) \rangle \equiv \int e^{ikx} \mathcal{D}(k) d^4k = \sum_{n=0}^{\infty} \left(\frac{x^2}{4} \right)^n \frac{1}{n!(n+1)!} \int (-k^2)^n \mathcal{D}(k) d^4k. \quad (81)$$

Note, that existence of $\langle \varphi(0)\varphi(0) \rangle$ implies that $\mathcal{D}(k)$, the nonperturbative part of the propagator, vanishes for large momenta faster than k^{-5} , while existence of $\langle \varphi(0)\partial^2\varphi(0) \rangle$ implies that $\mathcal{D}(k) < k^{-7}$, etc. All matrix elements $\langle \varphi(0)(\partial^2)^n\varphi(0) \rangle$ exist only if, for large momenta, the function $\mathcal{D}(k)$ vanishes faster than any power of $1/k^2$, say, like an exponential. Thus, the function $\mathcal{D}(k)$ is concentrated in the region of small momenta. This corresponds to the basic assumption that at large momenta one should enjoy the asymptotic freedom.

In other words, the exact propagator is represented as a sum of the perturbative $O(1/k^2)$ part and the nonperturbative (“condensate”) part that vanishes fast when the momentum k increases. Using such a decomposition, we obtain a modified diagram technique for the correlator functions. Consider, *e.g.*, the lowest order diagram corresponding to the product of two propagators. After the decomposition it produces four diagrams (see Fig.3), where the dashed lines correspond to the nonperturbative part of the propagators.

The first term (Fig.3a) is just the ordinary purely perturbative diagram having $\sim \log(q^2)$ behavior for large q^2 :

$$\int \frac{d^4k}{k^2(q-k)^2} \sim \log(q^2).$$

Two other diagrams (Figs.3b,c), as we will see below, behave like $1/q^2$. This behavior is completely determined by that of the perturbative propagator: the

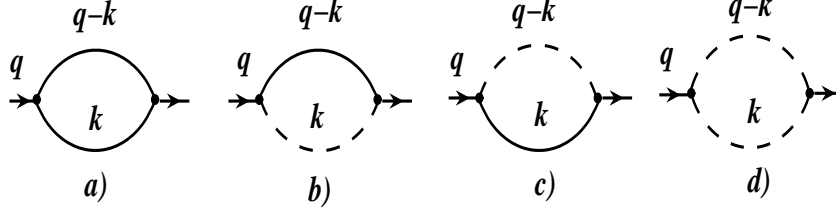


Figure 3: Operator product expansion for scalar one-loop diagram.

whole momentum flows through the perturbative line where its propagation is not suppressed. In the last diagram (Fig.3d), all possible momentum flows are strongly suppressed, *e.g.*, if $\mathcal{D}(k)$, the nonperturbative part of the propagator, vanishes exponentially at large k , so does this contribution also.

Now, let us analyze the contribution of the second diagram:

$$I^{\langle\varphi\varphi\rangle}(q^2) = \int \mathcal{D}(k^2) \frac{d^4 k}{(q-k)^2}. \quad (82)$$

As we discussed earlier, the function $\mathcal{D}(k^2)$ is concentrated in the region of small momenta. So it makes sense to expand the perturbative propagator $1/(q-k)^2$ in k :

$$\frac{1}{(q-k)^2} = \frac{1}{q^2} \sum_{n=0}^{\infty} \frac{2^n \{qk\}^{(n)}}{(q^2)^n} \theta(k^2 < q^2) + \frac{1}{k^2} \sum_{n=0}^{\infty} \frac{2^n \{qk\}^{(n)}}{(k^2)^n} \theta(q^2 < k^2) \quad (83)$$

where

$$\{qk\}^{(n)} \equiv \{q_{\mu_1} \dots q_{\mu_n}\} \{k^{\mu_1} \dots k^{\mu_n}\}$$

is the product of two symmetric

$$O^{\dots\mu_i\dots\mu_j\dots} = O^{\dots\mu_j\dots\mu_i\dots}$$

and traceless

$$O^{\dots\mu_i\dots\mu_j\dots} g_{\mu_i\mu_j} = 0$$

tensors formed from the vectors q and k , respectively. The symmetric-traceless tensors have a very useful property:

$$\int F(k^2) \{k^{\mu_1} \dots k^{\mu_n}\} d^4 k = 0$$

for $n \neq 0$, because it is impossible to construct a nontrivial symmetric-traceless combination using only the metric tensors $g_{\mu_i \mu_j}$. To do this, one should have at least one vector, *e.g.*,

$$\{p_{\mu_1} p_{\mu_2}\} = p_{\mu_1} p_{\mu_2} - \frac{1}{4} p^2 g_{\mu_1 \mu_2}.$$

Thus, in our case

$$\begin{aligned} I^{\langle \varphi \varphi \rangle}(p^2) &\equiv \int \mathcal{D}(k^2) \frac{d^4 k}{(q-k)^2} \\ &= \frac{1}{q^2} \int \mathcal{D}(k^2) \theta(k^2 < q^2) d^4 k + \int \mathcal{D}(k^2) \frac{d^4 k}{k^2} \theta(q^2 < k^2) \\ &= \frac{1}{q^2} \langle \varphi \varphi \rangle + \text{“quickly vanishing contribution”}, \end{aligned} \quad (84)$$

where the “quickly vanishing” term is given by the integral over large momenta:

$$\text{“quickly vanishing contribution”} = \int \left(\frac{1}{k^2} - \frac{1}{q^2} \right) \mathcal{D}(k^2) \theta(k^2 > q^2) d^4 k$$

where the function $\mathcal{D}(k^2)$ is strongly suppressed.

Thus, it is very easy to calculate this contribution: the whole external momentum flows through the perturbative line, so one should just multiply the perturbative propagator $1/q^2$ by the condensate $\langle \varphi \varphi \rangle$ factor. One can wonder, however, why the higher condensates (containing the ∂^2) did not appear? To answer this question, it is useful to analyze this contribution in the configuration representation. The original diagram decomposition now looks like

$$\begin{aligned} \mathcal{D}(x, 0) \mathcal{D}(0, x) &= \mathcal{D}^{pert}(x, 0) \mathcal{D}^{pert}(0, x) + \mathcal{D}^{pert}(x, 0) \langle \varphi(0) \varphi(x) \rangle \\ &\quad + \mathcal{D}^{pert}(0, x) \langle \varphi(x) \varphi(0) \rangle + \langle \varphi(0) \varphi(x) \rangle \langle \varphi(x) \varphi(0) \rangle, \end{aligned} \quad (85)$$

and its second term can be written (modulo trivial factors like 2π) as

$$\langle \varphi(0) \varphi(x) \rangle / x^2.$$

Now, using the x^2 -expansion for the $\langle \varphi(0) \varphi(x) \rangle$ - factor, we obtain

$$\frac{1}{x^2} \langle \varphi(0) \varphi(x) \rangle = \frac{1}{x^2} \langle \varphi \varphi \rangle + \frac{1}{x^2} \frac{x^2}{8} \langle \varphi(0) \varphi(x) \rangle + O(x^2) + \dots \quad (86)$$

In the momentum representation, the first term just corresponds to the $\langle \varphi \varphi \rangle / q^2$ contribution, while the second is proportional to $\delta^4(q)$ and can be discarded

since the external momentum q is assumed to be large. Higher terms are proportional to the derivatives of $\delta^4(q)$, and also can be ignored for large q . It is easy to realize, that the sum of all these terms just produces the “quickly vanishing contribution” discussed above. Only the terms singular for $x^2 = 0$ (in the configuration representation) can produce a power-like behavior in the momentum representation.

The expansion we discussed above is a simplified illustration of the operator product expansion¹²

$$T(j(x)j(0)) = \sum_{n=0} C_n(x^2)O_n(0). \quad (87)$$

It represents the T -product of two currents $j(x), j(0)$ as a sum of the local (composite) operators of increasing dimension (in units of mass). The coefficient functions $C_n(x^2)$ characterizing the contribution of a particular operator are ordered in their singularity at $x^2 = 0$. From a simple dimensional analysis it follows that there should be a simple relation between singularity of the coefficient function and the dimension of the relevant composite operator; *e.g.*, if

$$C_n(x^2) \sim \frac{1}{(x^2)^{N_0-n}}$$

then

$$O_n \sim (\mu^2)^{d_0+n},$$

so that after the Fourier transform one arrives at an expansion in powers of μ^2/q^2 . There are, of course, more complicated contributions into the OPE, corresponding to more complicated original diagrams.

Similar decomposition of propagators into perturbative and nonperturbative parts can be also performed in the QCD case. At this level of discussion, the only difference is that in QCD one has to deal with two types of fields: quarks q, \bar{q} and gluons A . Consider the decomposition of a two-loop diagram (see Fig.4).

- The first diagram (Fig.4a) corresponds to the purely perturbative contribution (unity operator in the OPE language),
- the second diagram (Fig.4b) (in fact there are two such diagrams) corresponds to the $\langle \bar{q}Aq \rangle$ -type contribution,
- the third (Fig.4c) is evidently proportional to the $\langle AA \rangle$ vacuum average, accompanied by the one-loop coefficient function,
- the fourth (Fig.4d) diagram (there is also another such diagram) brings in the four-quark vacuum average $\langle \bar{q}q\bar{q}q \rangle$.

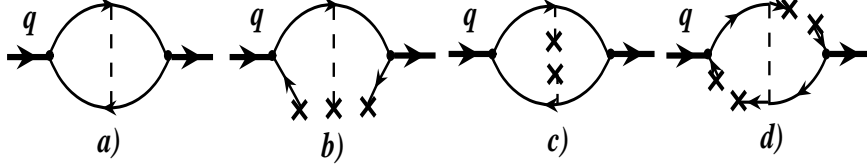


Figure 4: Operator product expansion for QCD two-loop diagram.

4 Operator Product Expansion: Gauge Invariance and Related Problems

Looking at the operators discussed in the previous section, one can notice that some of them, *e.g.*, $\langle \bar{q} \partial_\mu q \rangle$, $\langle \bar{q} A_\mu q \rangle$, $\langle A_\mu A_\nu \rangle$, etc., do not satisfy a very important requirement: they are not gauge invariant. And the product of gauge invariant operators in a gauge theory should have an expansion in gauge invariant operators like $\langle \bar{q} \hat{D}_\mu q \rangle$ and $\langle G_{\mu\nu}^a G^{a\mu\nu} \rangle$.

4.1 Notations

Let us introduce now the necessary notations. To begin with,

$$\hat{D}_\mu = \partial_\mu - ig \hat{A}_\mu$$

is the covariant derivative in the quark (fundamental) representation of the $SU(3)_c$ gauge group:

$$\hat{A}_\mu = A_\mu^a \tau^a,$$

with matrices $(\tau^a)_{AB}$ simply related to the Gell-Mann (3×3) λ -matrices

$$\tau^a \equiv \lambda^a / 2.$$

The indices A, B correspond to 3 possible quark colors: $A, B = 1, 2, 3$. The gluons have 8 colors described by the a -indices: $a = 1, \dots, 8$. The covariant derivative

$$\tilde{D}_\mu = \partial_\mu - ig \tilde{A}_\mu$$

acting on the gluonic field contains the (8×8) σ -matrices

$$\tilde{A}_\mu = A_\mu^a \sigma^a$$

characteristic for the gluonic (adjoint) representation of the $SU(3)$ color group:

$$(\sigma^a)_{bc} = -if_{abc},$$

where f_{abc} are the structure constants of the group:

$$[\tau^a, \tau^b] = if^{abc}\tau^c.$$

The gluonic field strength tensor

$$G_{\mu\nu}^a = \partial_\mu A_\nu - \partial_\nu A_\mu + gf^{abc}A_\mu^b A_\nu^c$$

can be related to the commutator of the covariant derivatives:

$$\hat{G}_{\mu\nu} = \frac{1}{ig}[\hat{D}_\mu, \hat{D}_\nu] \quad (88)$$

$$\tilde{G}_{\mu\nu} = \frac{1}{ig}[\tilde{D}_\mu, \tilde{D}_\nu]. \quad (89)$$

4.2 External field method

A rather innocent-sounding statement that only gauge invariant operators should appear in the operator product expansion implies that there are numerous nontrivial correlations between the coefficient functions of non-gauge-invariant operators originating from completely different diagrams. For example, the operator $\langle \bar{q}\partial_\mu q \rangle$ appears when one expands the nonlocal operator $\langle \bar{q}(0)q(x) \rangle$ in the simplest combination (see Fig.5a)

$$\langle \bar{q}(0)S(-x)q(x) \rangle,$$

while the operator $\langle \bar{q}\hat{A}_\mu q \rangle$ necessary to form a gauge-invariant sum, results from a more complicated combination (see Fig.5b)

$$\int \langle \bar{q}(0)S(-y)\gamma^\mu(ig)\hat{A}_\mu(y)S(y-x)q(x) \rangle d^4y.$$

In both cases, the perturbative propagators (in configuration space) are written explicitly, while the nonperturbative parts, in accordance with our previous discussions, are denoted by vacuum averages of the relevant normal products of fields. By a straightforward integration one can establish that

$$\int S(-y)\gamma^\mu S(y-x) d^4y = -x^\mu S(-x),$$

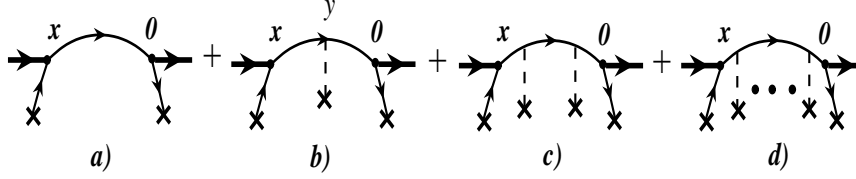


Figure 5: Summation over gluons inserted into hard quark propagator.

and, hence, the coefficient functions of the two operators are just those necessary to form the gauge invariant combination.

However, this is definitely not the method one would like to use in more complicated situations. In fact, it is much easier to analyze the sum of all the combinations differing only by the number of the gluons going into vacuum (see Fig.5).

One can write it as^{13,14} $\langle \bar{q}(0)S(0, x; A)q(x) \rangle$, where $S(0, x; A)$ has the meaning of the perturbative quark propagator in the (external) vacuum gluonic field A . It satisfies the Dirac equation

$$\left[i\gamma^\mu \left(\frac{\partial}{\partial x^\mu} - ig\hat{A}_\mu(x) \right) - m \right] S(x, y; A) = -\delta^4(x - y). \quad (90)$$

Solving it by using perturbation theory in A , one gets back the original diagrammatic expansion.

The function $S(x, y; A)$ is not gauge invariant. From the good old electrodynamics it is known that propagation of a charged particle in the external field results in a phase factor given by the exponential of a line integral. In the QCD case it looks like

$$\hat{P}(x, y; A; \mathcal{C}) = P \exp \left(ig \int_{\mathcal{C}} \hat{A}_\mu(z) dz^\mu \right),$$

where P means that the color matrices should be ordered along the path \mathcal{C} connecting the points x and y . Furthermore, it is known that the combination

$$\langle \bar{q}(y) \hat{P}(y, x; A; \mathcal{C}) q(x) \rangle$$

is gauge invariant (though path-dependent). This gives the idea how one can construct the operator product expansion in an explicitly gauge invariant way. Let us try the ansatz

$$S(x, y; A) = \hat{E}(x, y; A) S(x, y; A) \quad (91)$$

where $\hat{E}(x, y; A)$ is the P -exponential for the straight-line path connecting the points x and y :

$$\hat{E}(x, y; A) = P \exp \left[ig(x^\mu - y^\mu) \int_0^1 \hat{A}_\mu(z) dt \right]_{z=y+t(x-y)}.$$

It is straightforward to derive that the original Dirac equation is satisfied only if the function $\mathcal{S}(x, y; A)$ is a solution to the modified Dirac equation

$$\left[i\gamma^\mu \left(\frac{\partial}{\partial x^\mu} - ig\hat{A}_\mu(x; y) \right) - m \right] \mathcal{S}(x, y; A) = -\delta^4(x - y) \quad (92)$$

that differs from the original one only by the change

$$A_\mu^a(x) \rightarrow \mathcal{A}_\mu^a(x; y) = (x^\nu - y^\nu) \int_0^1 G_{\nu\mu}^b(z) \tilde{E}^{ba}(z, y) t dt. \quad (93)$$

Here, $z = y + t(x - y)$ and \tilde{E} is a straight-line-ordered exponential with the A -fields multiplied by the σ -matrices of the gluonic representation. It appears after one commutes the \hat{E} -exponentials through a τ -matrix:

$$(\tau^a)_{AB} \hat{E}_{BC}(z, y) = \hat{E}_{AB}(z, y) (\tau^b)_{BC} \tilde{E}^{ab}(z, y). \quad (94)$$

The commutation rule is based on the well-known formula

$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!} [A, [A, B]] + \dots \quad (95)$$

and the relation

$$[\tau_a, \tau_b] = -(\sigma_a)_{bc} \tau_c. \quad (96)$$

Surprisingly enough, the presence of the \tilde{E} -factor only simplifies the situation, because the derivatives in the Taylor expansion of the $G\tilde{E}$ -combination are just the covariant ones:

$$G_{\nu\mu}^b(z) \tilde{E}^{ba}(z, y) = \sum_{n=0}^{\infty} \frac{1}{n!} G_{\nu\mu; \mu_1 \dots \mu_n}^a(y) (z - y)^{\mu_1} \dots (z - y)^{\mu_n}. \quad (97)$$

As a result, the new field $\mathcal{A}_\mu^a(x; y)$ can be expressed in terms of the gluon field stress tensor $G_{\mu\nu}$ and its covariant derivatives (taken, of course, in the gluonic representation):

$$\mathcal{A}_\mu^a(x; y) = \sum_{n=0}^{\infty} \frac{1}{(n+2)n!} (x - y)^\nu (x - y)^{\mu_1} \dots (x - y)^{\mu_n} G_{\nu\mu; \mu_1 \dots \mu_n}^a(y). \quad (98)$$

This means that the function $\mathcal{S}(x, y; A)$ depends on the gluonic field only through the gluon field stress tensor $G_{\mu\nu}$ and its covariant derivatives. Solving the equation for $\mathcal{S}(x, y; A)$ perturbatively, one obtains the $g\mathcal{A}$ -expansion that has the same structure as the original gA -expansion. A very important difference is that the resulting operators have an explicitly gauge invariant form. For example, taking the lowest (zero) order term one obtains the contribution

$$\langle \bar{q}(y) S(y-x) \hat{E}(y, x; A) q(x) \rangle \quad (99)$$

containing the gauge-invariant operator

$$\langle \bar{q}(y) \hat{E}(y, x; A) q(x) \rangle = \sum_{n=0}^{\infty} \frac{1}{n!} (x-y)_{\mu_1} \dots (x-y)_{\mu_n} \langle \bar{q} \hat{D}^{\mu_1} \dots \hat{D}^{\mu_n} q \rangle|_y. \quad (100)$$

We used the direct straight-line path from x to y in our ansatz. In general, it is more convenient to use the path formed by two straight lines: from x to some fixed point z_0 (usually taken at the origin, $z_0 = 0$) and then from z_0 to y . The only change then will be that quark and gluon fields $q(\xi)$, $A(\xi)$ will be finally accompanied by the $E(\xi, z_0; A)$ exponentials, so that a covariant expansion will be obtained if one expands at the fixed point z_0 . Using translation invariance, one can always substitute matrix elements $\langle \bar{q}(z_0) \dots G(z_0) \dots q(z_0) \rangle$, resulting from such a procedure, by $\langle \bar{q}(0) \dots G(0) \dots q(0) \rangle$. Less trivial is the disappearance of the z_0 -dependence from the coefficient functions. One can enjoy it only if the calculations had no errors, and after summation over all diagrams (of the modified expansion) producing the same operator. The latter observation suggests that z_0 works as a gauge parameter. Indeed, it is easy to notice that the relevant P -exponentials

$$E(x, z_0; a) = P \exp[i g (x^\mu - z_0^\mu) \int_0^1 A_\mu(z) dt] |_{z=z_0+t(x-z_0)} \quad (101)$$

disappear (are equal to 1) if one imposes the gauge condition^{15,16,17,18,14}

$$(x^\mu - z_0^\mu) A_\mu(x) = 0. \quad (102)$$

Furthermore, the “calligraphic” field \mathcal{A} in this (Fock-Schwinger or fixed point) gauge coincides with the ordinary one $A = \mathcal{A}(\{G\})$, *i.e.*, in this gauge not only $G_{\mu\nu}$ can be expressed in terms of A_μ , but also the vector-potential A_μ can be represented as a (nonlocal) functional of the field strength tensor $G_{\mu\nu}$. This means that the Fock-Schwinger gauge is a physical gauge. Another example is the axial gauge¹⁹

$$n_\mu A^\mu(x) = 0 \quad (103)$$

with n_μ being a fixed vector. In this gauge one can also express¹⁹ A_μ in terms of $G_{\mu\nu}$:

$$A_\mu(x) = n^\nu \int_0^\infty G_{\nu\mu}(x + tn) dt. \quad (104)$$

Imposing the Fock-Schwinger gauge condition, one can forget about the exponentials. One should remember, however, that the “ordinary” derivatives in this gauge should be treated as the covariant ones, *e.g.*, they do not commute, etc. To illustrate the typical tricks and conventions, let us consider the simplest gauge invariant nonlocal condensate

$$\begin{aligned} & \langle \bar{q}(0) \hat{E}(0, x; A) q(x) \rangle|_{(xA)=0} = \langle \bar{q}(0) q(x) \rangle \\ & = \sum_{n=0}^{\infty} \frac{1}{n!} x_{\mu_1} \dots x_{\mu_n} \langle \bar{q} \hat{D}^{\mu_1} \dots \hat{D}^{\mu_n} q \rangle = \langle \bar{q} q \rangle + \frac{x^2}{8} \langle \bar{q} \hat{D}^2 q \rangle + \dots \end{aligned} \quad (105)$$

The ratio

$$\lambda^2 \equiv \frac{\langle \bar{q} \hat{D}^2 q \rangle}{\langle \bar{q} q \rangle} \quad (106)$$

can be interpreted as the average virtuality of the vacuum quarks. It is possible, however, to represent the $\langle \bar{q} \hat{D}^2 q \rangle$ operator in a different form:

$$\begin{aligned} \langle \bar{q} \hat{D}^2 q \rangle & \equiv \langle \bar{q} \hat{D}^\mu \hat{D}_\mu q \rangle = \langle \bar{q} \hat{D}^\mu \hat{D}^\nu q \rangle g_{\mu\nu} = \langle \bar{q} \hat{\mathcal{D}} \hat{\mathcal{D}} q \rangle - \langle \bar{q} \hat{D}^\mu \hat{D}^\nu \sigma_{\mu\nu} q \rangle \\ & = -m_q^2 \langle \bar{q} q \rangle - \frac{1}{2} \langle \bar{q} [\hat{D}^\mu, \hat{D}^\nu] \sigma_{\mu\nu} q \rangle = -m_q^2 \langle \bar{q} q \rangle + \frac{1}{2} \langle \bar{q} i g \hat{G}^{\mu\nu} \sigma_{\mu\nu} q \rangle \end{aligned} \quad (107)$$

where we have used the identity

$$g_{\mu\nu} = \gamma_\mu \gamma_\nu - \frac{\gamma_\mu \gamma_\nu - \gamma_\nu \gamma_\mu}{2} = \gamma_\mu \gamma_\nu - \sigma_{\mu\nu},$$

the equation of motion

$$\hat{\mathcal{D}} q(x) = -i m_q q(x) \quad (108)$$

and the definition of the field strength tensor

$$[\hat{D}_\mu, \hat{D}_\nu] = -i g \hat{G}_{\mu\nu}. \quad (109)$$

Thus, the “average virtuality” of the vacuum quarks $\langle \bar{q} \hat{D}^2 q \rangle$ is directly related to the “average vacuum gluonic field strength” $\langle \bar{q} i g \hat{G}^{\mu\nu} \sigma_{\mu\nu} q \rangle$. In many papers one can find the notation

$$\langle \bar{q} i g \hat{G}^{\mu\nu} \sigma_{\mu\nu} q \rangle \equiv m_0^2 \langle \bar{q} q \rangle. \quad (110)$$

Using it, one can write the relation

$$\lambda^2 = \frac{m_0^2}{2} - m_q^2. \quad (111)$$

For light quarks, the m_q^2 term can be neglected.

4.3 Equations of motion and the current conservation

The equation of motion $\hat{D}q(x) = -im_q q(x)$ brings in correlations between matrix elements of different operators. These correlations, in fact, are very important to preserve the symmetry properties required by the structure of the currents $j(x)$ in the correlators. For example, one might expect that a correlator of the electromagnetic vector currents

$$\Pi_{\mu\nu}(p) = \int \langle T j_\mu(0) j_\nu(x) \rangle e^{ipx} d^4x \quad (112)$$

should be transverse:

$$\Pi_{\mu\nu}(p) = (p_\mu p_\nu - g_{\mu\nu} p^2) \Pi(p^2) \quad (113)$$

due to the current conservation condition $\partial^\mu j_\mu = 0$. Indeed, for the perturbative one-loop contribution, using the explicit form of the propagator in the configuration space

$$S(x) \propto -2i \not{x} \left[\frac{1}{x^2} + \frac{m^2}{4} \right] + \frac{m}{x^2} + O(m^3) \quad (114)$$

one finds that the integrand is

$$\frac{1}{(x^2)^4} [g_{\mu\nu} x^2 - 2x_\mu x_\nu] - \frac{m^2}{(x^2)^3} [g_{\mu\nu} x^2 - 4x_\mu x_\nu] + \dots \quad (115)$$

It is easy to see that both terms satisfy the transversality condition

$$\frac{\partial}{\partial x^\mu} \{ \dots \} = 0,$$

and this means that

$$\Pi_{\mu\nu}^{pert}(p) = (p_\mu p_\nu - g_{\mu\nu} p^2) \Pi^{pert}(p^2)$$

at least up to the third order in m . One can, in fact, perform the calculations in the momentum space directly, to see that this property of $\Pi_{\mu\nu}^{pert}(p)$ is valid to all orders in m .

Now, let us consider the simplest nonperturbative contribution. In the momentum representation it looks like

$$\langle \bar{q} \gamma_\mu \frac{\not{p} + m}{p^2 - m^2} \gamma_\nu q \rangle \sim \frac{m g_{\mu\nu}}{p^2} \langle \bar{q} q \rangle. \quad (116)$$

However, it seems that there is no way to get the $p_\mu p_\nu$ term, since there is only one \not{p} term in the numerator. To solve this puzzle, let us consider this contribution in the configuration representation. Up to $O(m^2)$ -terms one has

$$\begin{aligned} \langle \bar{q}(0) \gamma_\mu S(x) \gamma_\nu q(x) \rangle &\propto \langle \bar{q}(0) \gamma_\mu \left(-2i \frac{\not{x}}{(x^2)^2} + \frac{m}{x^2} \right) \gamma_\nu q(x) \rangle + O(m^2) \\ &= -\frac{2i}{(x^2)^2} \langle \bar{q}(0) \gamma_\mu \not{x} \gamma_\nu q(x) \rangle + \frac{m}{x^2} \langle \bar{q}(0) \gamma_\mu \gamma_\nu q(x) \rangle + O(m^2). \end{aligned} \quad (117)$$

Note, that vacuum averages of local operators with an odd number of indices vanish, and only those with an even number survive. Incorporating the Lorentz invariance, we obtain

$$\begin{aligned} \langle \bar{q}(0) \gamma_\mu \gamma_\nu q(0) \rangle &= g_{\mu\nu} \langle \bar{q} q \rangle, \\ \langle \bar{q}(0) \gamma_\mu \gamma_\alpha \gamma_\nu q(0) \rangle &= 0, \\ \langle \bar{q}(0) \gamma_\mu \gamma_\alpha \gamma_\nu D_\beta q(0) \rangle &= (g_{\mu\alpha} g_{\nu\beta} + g_{\nu\alpha} g_{\mu\beta} - g_{\mu\nu} g_{\alpha\beta}) \langle \bar{q} \gamma_\sigma D_\beta q \rangle, \\ \langle \bar{q} \gamma_\sigma D_\beta q \rangle &= \frac{g_{\sigma\beta}}{4} \langle \bar{q} \hat{D} q \rangle. \end{aligned} \quad (118)$$

Finally, incorporating the equation of motion

$$\langle \bar{q} \hat{D} q \rangle = -im \langle \bar{q} q \rangle \quad (119)$$

one obtains the transverse result

$$\langle \bar{q}(0) \gamma_\mu S(x) \gamma_\nu q(x) \rangle \propto \frac{m}{2(x^2)^2} \langle \bar{q} q \rangle [g_{\mu\nu} x^2 + 2x_\mu x_\nu]. \quad (120)$$

It is instructive to analyze also the $\langle GG \rangle$ contribution. First, using the exponential representation, it is easy to see that for the closed loop the exponentials corresponding to the upper and the lower lines cancel each other, and only the $O(G)$ terms remain. This general result can be illustrated on the example of one-loop diagrams with two vacuum gluons. To study, how the cancellation proceeds in this case, one should expand the exponential up to $O(A^2)$ terms:

$$\begin{aligned} S(x, A) &= S(x) \left\{ 1 + ig(xA) + \frac{(ig)^2}{2} (xA)^2 + \dots \right\}, \\ S(-x, A) &= S(-x) \left\{ 1 - ig(xA) + \frac{(ig)^2}{2} (xA)^2 + \dots \right\}. \end{aligned} \quad (121)$$

Now, combining the contributions of the three relevant diagrams, one obtains

$$(ig)^2 \left\{ \frac{1}{2}(xA)^2 + \frac{1}{2}(xA)^2 - (xA)^2 \right\} = 0, \quad (122)$$

i.e., the A -dependence disappeared. To calculate the $O(G)$ - terms, it is most convenient to use the Fock-Schwinger gauge $(xA) = 0$, in which

$$\begin{aligned} A_\mu(x) &= x^\nu \int_0^1 G_{\nu\mu}(tx) t dt = \sum_{n=0}^{\infty} \frac{x^\nu x^{\mu_1} \dots x^{\mu_n}}{(n+2)n!} G_{\nu\mu;\mu_1\dots\mu_n}(0) \\ &= \frac{1}{2} x^\nu G_{\nu\mu}(0) + \dots \end{aligned} \quad (123)$$

Using this representation, one can construct the Feynman rules for the vertices corresponding to $G_{\mu\nu}$, $G_{\mu\nu;\mu_1}$, etc. For example, in the momentum representation

$$\frac{x_\nu}{2} G_{\nu\mu} \Rightarrow \frac{1}{2i} G_{\nu\mu} \frac{\partial}{\partial k_\nu}, \quad (124)$$

and the insertion of such a vertex into the quark propagator gives

$$\frac{1}{2i} \frac{\not{k}}{k^2} \gamma_\mu G_{\nu\mu} \frac{\partial}{\partial k_\nu} \frac{\not{k}}{k^2} = \frac{i}{2} G_{\nu\mu} \frac{\not{k}}{k^2} \gamma_\mu \frac{\not{k}}{k^2} \gamma_\nu \frac{\not{k}}{k^2}. \quad (125)$$

The result looks like an insertion of two A -fields, A_ν and A_μ , antisymmetrized with respect to the interchange of μ and ν . There is an important thing to remember while constructing the diagrammatic technique for the G -type vertices: in the momentum representation they contain derivatives acting onto all the propagators between the G -vertex and the Fock-Schwinger zero point. The G -inserted propagator looks simpler in the configuration space:

$$S(x, G) = -\frac{\not{x}}{2\pi^2(x^2)^2} - \frac{g}{16\pi^2 x^2} x_\alpha \epsilon_{\alpha\beta\mu\nu} \hat{G}^{\mu\nu} \gamma_\beta \gamma_5 + O(z_0) \langle GG \rangle + \dots \quad (126)$$

where z_0 is the Fock-Schwinger fixed point. If one takes $z_0 = 0$, then the $O(\langle GG \rangle)$ term disappears. This means that in the gauge $(xA) = 0$ only the diagram with the G -insertions into different lines survives. Its contribution in the configuration space is easily calculated by just multiplying the two $S(x, G)$ propagators. Taking the trace over the γ -matrices, one obtains the result

$$-\frac{2i}{3(16\pi^2)^2(x^2)^2} (g_{\mu\nu} x^2 + 2x_\mu x_\nu) \langle g^2 G^2 \rangle \quad (127)$$

proportional to the same transverse structure as the contribution due to the $m_q \langle \bar{q}q \rangle$ operator. This is because both operators have the same dimension, equal to 4 in mass units.

The QCD sum rules for the “standard” mesons, like $\rho, \pi, K, \text{etc.}$, are based on the OPE including the operators of dimension 6, at least. One of these operators can be easily obtained from the Taylor expansion of the nonlocal quark condensate containing a γ -matrix

$$\langle \bar{q}(0) \gamma_\sigma q(x) \rangle = \langle \bar{q} \gamma_\sigma \hat{D}_\alpha q \rangle x^\alpha + \frac{1}{3!} \langle \bar{q} \gamma_\sigma \hat{D}_{\alpha_1} \hat{D}_{\alpha_2} \hat{D}_{\alpha_3} q \rangle x^{\alpha_1} x^{\alpha_2} x^{\alpha_3} + \dots \quad (128)$$

We recall that the terms, containing an even number of derivatives, vanish since the total number of indices is odd in these cases. Incorporating the Lorentz invariance, we obtain

$$\begin{aligned} \langle \bar{q}(0) \gamma_\sigma q(x) \rangle &= \frac{x^\sigma}{4} \langle \bar{q} \hat{\not{D}} q \rangle + \frac{x^\sigma x^2}{144} (\langle \bar{q} \hat{\not{D}} \hat{D}^\alpha \hat{D}_\alpha q \rangle \\ &\quad + \langle \bar{q} \hat{D}^\alpha \hat{\not{D}} \hat{D}_\alpha q \rangle + \langle \bar{q} \hat{D}^\alpha \hat{D}_\alpha \hat{\not{D}} q \rangle). \end{aligned} \quad (129)$$

Next step is to use the translation invariance $\langle \partial^\mu \bar{q}(0) \dots q(0) \rangle = 0$, the equation of motion $\hat{\not{D}} q = -im_q q$ and the commutation relation $[\hat{D}_\alpha, \hat{D}_\mu] = ig \hat{G}_{\alpha\mu}$ to get

$$\langle \bar{q}(0) \gamma_\sigma q(x) \rangle = \frac{-im x^\sigma}{4} \left(\langle \bar{q} q \rangle + \frac{x^2}{12} \langle \bar{q} \hat{D}^2 q \rangle \right) + \frac{ig}{288} x^\sigma x^2 \langle \bar{q} \gamma_\mu [\hat{D}_\alpha, \hat{G}^{\mu\alpha}] q \rangle. \quad (130)$$

The commutator $[\hat{D}_\alpha, \hat{G}^{\mu\alpha}]$ is equal to the covariant derivative of the gluonic field

$$[\hat{D}_\alpha, \hat{G}^{\mu\alpha}] = \hat{G}^{\mu\alpha}_{;\alpha}. \quad (131)$$

The final step is to use the gluonic equation of motion

$$\hat{G}^{\mu\alpha}_{;\alpha} = j_a^\mu \tau^a \equiv \sum_{i=u,d,s} \langle \bar{q}_i \gamma^\mu \tau_a q_i \rangle \tau^a \quad (132)$$

analogous to the Maxwell equation. Thus, due to the equations of motion, the apparently two-quark operator produces a four-quark term:

$$\langle \bar{q}(0) \gamma_\sigma q(x) \rangle = \frac{-im x^\sigma}{4} (\langle \bar{q} q \rangle + \frac{x^2}{12} \langle \bar{q}(0) \hat{D}^2 q \rangle) + \frac{ig}{288} x^\sigma x^2 \langle \bar{q} \gamma_\mu j_a^\mu \tau^a q \rangle. \quad (133)$$

5 QCD Sum Rules for ρ -meson

5.1 Dispersion relation

As pointed out by SVZ (whose analysis¹ we closely follow in this section) we closely to determine the basic parameters characterizing the ρ -meson, *e.g.*, its mass m_ρ and its e^+e^- coupling constant g_ρ , one should consider the correlator of two vector currents

$$j_\mu^{(\rho)} = \frac{1}{2}(\bar{u}\gamma_\mu u - \bar{d}\gamma_\mu d) \quad (134)$$

with the ρ -meson quantum numbers:

$$\Pi_{\mu\nu}^{(\rho)}(q) = (q_\mu q_\nu - q^2 g_{\mu\nu})\Pi^{(\rho)}(q^2) = i \int e^{iqx} \langle 0 | T(j_\mu^{(\rho)}(x) j_\nu^{(\rho)}(0)) | 0 \rangle d^4x. \quad (135)$$

Via the dispersion relation

$$\Pi^{(\rho)}(q^2 = -Q^2) = \Pi^{(\rho)}(0) - \frac{Q^2}{12\pi^2} \int_0^\infty \frac{R^{I=1}(s)}{s(s+Q^2)} ds \quad (136)$$

this correlator is related to the total cross section of the e^+e^- -annihilation to hadrons with the isospin $I = 1$, measured in units of the $e^+e^- \rightarrow \mu^+\mu^-$ cross section :

$$R^{I=1} = \frac{\sigma(e^+e^- \rightarrow \text{hadrons}, I = 1)}{\sigma(e^+e^- \rightarrow \mu^+\mu^-)}. \quad (137)$$

Indeed, the $j_\mu^{(\rho)}$ coincides with the isovector part of the electromagnetic current and is responsible, therefore, for the production of the hadrons with $I = 1$ in the e^+e^- -annihilation. The correlator $\Pi_{\mu\nu}^{(\rho)}(q)$ contains, of course, only the hadronic part of the relevant factors, and that explains the division by $\sigma(e^+e^- \rightarrow \mu^+\mu^-)$.

5.2 Operator product expansion

On the other hand, one can construct the operator product expansion for the correlator $\Pi_{\mu\nu}^{(\rho)}(q)$ calculating the relevant Feynman diagrams. To begin with, one should calculate the perturbative diagrams shown in Fig.6 to extract the perturbative spectral density

$$\rho^{pert}(s) = \frac{3}{2} \theta(s) \left\{ 1 + \frac{\alpha_s}{\pi} \right\} \quad (138)$$

which is essentially the isovector ($I = 1$ isospin projection) quark charge squared $(\frac{1}{2})^2$ multiplied by 2 (u - and d - quarks) and multiplied by the number

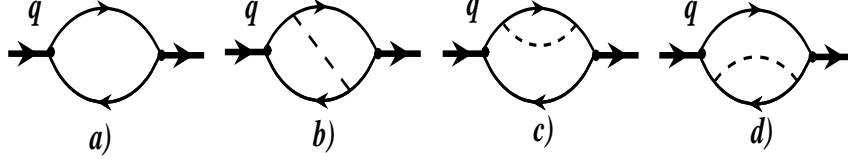


Figure 6: Perturbative contribution up to two loops.

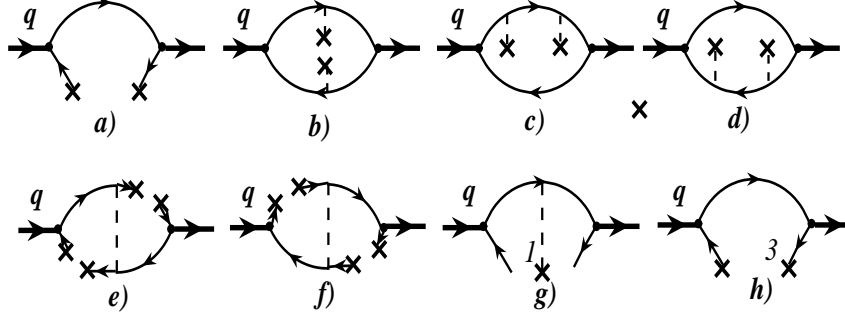


Figure 7: Nonperturbative contributions up to dimension six.

of colors (3). The simplest nonperturbative contribution is due to the quark condensate (see Fig.7a). It is proportional to the current quark masses:

$$\Pi^{\bar{q}q} \sim \frac{\langle 0 | m_u \bar{u}u + m_d \bar{d}d | 0 \rangle}{(Q^2)^2},$$

with $m_u \sim 4MeV, m_d \sim 7MeV$. Hence, this contribution is rather small numerically. The contribution

$$\Pi^{GG} \sim g^2 \frac{\langle 0 | GG | 0 \rangle}{(Q^2)^2}$$

due to the gluon condensate is given by three diagrams shown in Fig.7b-d. As mentioned at the end of the previous section, only the diagram 7b contributes in the Fock-Schwinger gauge $x_\mu A^\mu(x) = 0$. It is rather straightforward to

calculate the diagrams 7e,f proportional to

$$\frac{g^2 \langle \bar{q} q \bar{q} q \rangle}{(Q^2)^3}.$$

However, the contributions of the same four-quark type can be obtained also from the diagrams 7g,h which originally are bilinear in quark fields. Notice, however, the numbers 1 and 3 attached to the relevant lines going into the vacuum. They indicate how many covariant derivatives are acting on the relevant “vacuum” field. Thus, the matrix elements are $\langle \bar{q} \gamma_\mu \tilde{D}_\nu \hat{G}_{\alpha\beta} q \rangle$ and $\langle \bar{q} \gamma_\mu \hat{D}_{\nu_1} \hat{D}_{\nu_2} \hat{D}_{\nu_3} q \rangle$. As discussed at the end of Section 4, after vacuum averaging and commutations only the $\langle \bar{q} \gamma_\mu \tilde{D}_\nu \hat{G}_{\mu\nu} q \rangle$ component survives and, by equations of motion it is reduced to $\langle 0 | \bar{q} \gamma_\mu \tau^a j_\mu^a q | 0 \rangle$ with

$$j_\mu^a = \sum_{i=u,d,s} \langle 0 | \bar{q}_i \gamma_\mu \tau^a q_i | 0 \rangle.$$

5.3 Basic QCD sum rule for the ρ -meson channel

After the Borel transformation that modifies the dispersion integral

$$\int \frac{\rho(s) ds}{s + Q^2} \rightarrow \int e^{-s/M^2} \rho(s) \frac{ds}{M^2} \quad (139)$$

and the power corrections

$$\frac{1}{(Q^2)^n} \rightarrow \frac{1}{(n-1)!(M^2)^n} \quad (140)$$

one obtains the following QCD sum rule¹:

$$\begin{aligned} \int_0^\infty e^{-s/M^2} R^{I=1}(s) ds &= \frac{3}{2} M^2 \left[1 + \frac{\alpha_s(M)}{\pi} + \frac{4\pi^2}{M^4} (\langle m_u \bar{u} u \rangle + \langle m_d \bar{d} d \rangle) + \right. \\ &+ \frac{\pi^2}{3M^4} \langle \frac{\alpha_s}{\pi} GG \rangle - \frac{2\pi^3}{M^6} \langle \alpha_s (\bar{u} \gamma_\alpha \gamma_5 \tau^a u - \bar{d} \gamma_\alpha \gamma_5 \tau^a d)^2 \rangle - \\ &\left. - \frac{4\pi^3}{9M^6} \langle \alpha_s (\bar{u} \gamma_\alpha \tau^a u + \bar{d} \gamma_\alpha \tau^a d) \sum_{i=u,d,s} \langle 0 | \bar{q}_i \gamma_\mu \tau^a q_i | 0 \rangle \right] \quad (141) \end{aligned}$$

It is the starting point for further analysis.

5.4 Fixing the condensates

Next step is to specify the numerical values of the matrix elements on the r.h.s. of this sum rule (the same values will be used in all other sum rules).

- The first matrix element is in fact fixed by the current algebra:

$$\langle m_u \bar{u}u \rangle + \langle m_d \bar{d}d \rangle = -\frac{1}{2} f_\pi^2 m_\pi^2 = -1.7 \cdot 10^{-4} \text{GeV}^4 \quad (142)$$

- The value of the gluonic condensate was extracted from the analysis of the QCD charmonium sum rules¹

$$\langle \frac{\alpha_s}{\pi} GG \rangle = 1.2 \cdot 10^{-2} \text{GeV}^4. \quad (143)$$

- The four-quark operators are first approximated by the value dictated by the vacuum dominance (or factorization) hypothesis:

$$\langle \bar{\psi}_{\{A\}} \psi_{\{B\}} \bar{\psi}_{\{C\}} \psi_{\{D\}} \rangle = \langle \bar{\psi}_{\{A\}} \psi_{\{B\}} \rangle \langle \bar{\psi}_{\{C\}} \psi_{\{D\}} \rangle - \langle \bar{\psi}_{\{A\}} \psi_{\{D\}} \rangle \langle \bar{\psi}_{\{C\}} \psi_{\{B\}} \rangle \quad (144)$$

where the subscripts A, B, C, D include spin, color and flavour. Using the orthogonality relation

$$\langle \bar{\psi}_{\{A\}} \psi_{\{B\}} \rangle = \frac{\delta_{AB}}{N} \langle \bar{\psi} \psi \rangle \quad (145)$$

one can reduce then any four quark operator to the $\langle \bar{q}q \rangle^2$ form. In our case

$$\begin{aligned} \alpha_s \langle (\bar{u} \gamma_\alpha \gamma_5 \tau^a u - \bar{d} \gamma_\alpha \gamma_5 \tau^a d)^2 \rangle |_{\text{vacuum dominance}} &= \\ &= \frac{32}{9} \alpha_s \langle \bar{q}q \rangle^2 \approx 6.5 \cdot 10^{-4} \text{GeV}^6 \end{aligned} \quad (146)$$

$$\begin{aligned} \alpha_s \langle (\bar{d} \gamma_\alpha \tau^a d) \sum_{i=u,d,s} \langle \bar{q}_i \gamma_\alpha \tau^a q_i \rangle \rangle |_{\text{vacuum dominance}} &= \\ &= -\frac{32}{9} \alpha_s \langle \bar{q}q \rangle^2 \approx -6.5 \cdot 10^{-4} \text{GeV}^6, \end{aligned} \quad (147)$$

with the numerical values following from the estimate

$$\alpha_s \langle \bar{q}q \rangle^2 \approx -1.8 \cdot 10^{-4} \text{GeV}^6 \quad (148)$$

which can be justified using the above relation between the quark condensate, quark masses ($m_u + m_d = 11$ MeV, experimental values of the pion mass and pion decay constant f_π , under the assumption that the relevant normalization point corresponds to $\alpha_s = 0.7$. In fact, this estimate amounts to fixing one of the most important input parameters of the QCD sum rule method and the quoted value is supported by successful predictions for many different resonances.

5.5 Qualitative analysis of the QCD sum rule

Using the vacuum dominance hypothesis, we can rewrite the sum rule in a more compact form:

$$\int_0^\infty e^{-s/M^2} R^{I=1}(s) ds = \frac{3}{2} M^2 \left[1 + \frac{\alpha_s(M)}{\pi} - \frac{2\pi^2 f_\pi^2 m_\pi^2}{M^4} + \frac{\pi^2}{3M^4} \left\langle \frac{\alpha_s}{\pi} GG \right\rangle - \frac{448\pi^3}{M^6} \alpha_s \langle \bar{q}q \rangle^2 \right]. \quad (149)$$

Substituting the numerical values of the condensate parameters gives

$$\int_0^\infty e^{-s/M^2} R^{I=1}(s) ds = \frac{3}{2} M^2 \left[1 + \frac{\alpha_s(M)}{\pi} + 0.1 \left(\frac{0.6 \text{ GeV}^2}{M^2} \right)^2 - 0.14 \left(\frac{0.6 \text{ GeV}^2}{M^2} \right)^3 \right]. \quad (150)$$

Our experience with the sum rules for the quantum-mechanical oscillator (see Section 2) suggests that to calculate the ρ -meson mass we should incorporate also the “daughter” sum rule resulting from the original one after the differentiation with respect to $1/M^2$:

$$\int_0^\infty e^{-s/M^2} s R^{I=1}(s) ds = \frac{3}{2} M^4 \left[1 + \frac{\alpha_s(M)}{\pi} - 0.1 \left(\frac{0.6 \text{ GeV}^2}{M^2} \right)^2 + 0.28 \left(\frac{0.6 \text{ GeV}^2}{M^2} \right)^3 \right]. \quad (151)$$

Note now that power corrections in the sum rules above are small for the values of the Borel parameter M as low as $M^2 = m_\rho^2 = 0.6 \text{ GeV}^2$:

$$\int_0^\infty e^{-s/m_\rho^2} R^{I=1}(s) ds = \frac{3}{2} m_\rho^2 [1 + 0.1 + 0.1 - 0.14], \quad (152)$$

$$\int_0^\infty e^{-s/m_\rho^2} s R^{I=1}(s) ds = \frac{3}{2} m_\rho^4 [1 + 0.1 - 0.1 + 0.28]. \quad (153)$$

At such a low value of M^2 one should expect that the integral over the physical cross section is dominated by the lowest resonance, *i.e.*, by the ρ -meson. Its contribution to $R^{I=1}(s)$, in the limit of the vanishing width, can be written as

$$R_\rho^{I=1}(s) = \frac{12\pi^2 m_\rho^2}{g_\rho^2} \delta(s - m_\rho^2). \quad (154)$$

These observations allow one to make a rough estimate of the predictions following from the QCD sum rule. To this end we, following SVZ¹

- neglect, for $M^2 = m_\rho^2$, the power corrections in the r.h.s. of the sum rules,

- neglect, for $M^2 = m_\rho^2$, the higher state contributions in the l.h.s. This gives the following relation

$$\frac{12\pi^2 m_\rho^2}{g_\rho^2} e^{-1} \approx \frac{3}{2} m_\rho^2. \quad (155)$$

Simplifying it results in a famous SVZ¹ expression for the g_ρ constant in terms of fundamental constants π and e :

$$\frac{g_\rho^2}{4\pi} \cong \frac{2\pi}{e} \approx 2.3, \quad (156)$$

to be compared with the experimental value 2.36 ± 0.18 .

5.6 Fitting the spectrum parameters

The standard procedure is to extract the parameters of the hadronic spectrum requiring the best agreement between the two sides of the sum rule. Of course, having only a few first terms of the operator product expansion one should not hope to reproduce all the details of the resonance structure in a particular channel. Only the gross features of the spectrum can be extracted from the QCD sum rules. So, let us take the ansatz “first narrow resonance plus continuum” which has been discussed in the oscillator case. In fact, one should expect that in QCD this ansatz should work better, since the higher hadronic resonances are very broad, and the total cross section of their production is known to be well reproduced by the parton model (*i.e.*, perturbative) calculation. Specifically, we take

$$R_\rho^{I=1}(s) = \frac{12\pi^2 m_\rho^2}{g_\rho^2} \delta(s - m_\rho^2) + \frac{3}{2} \left(1 + \frac{\alpha_s}{\pi}\right) \theta(s - s_0). \quad (157)$$

The parameters to fit are the coupling constant g_ρ^2 , the ρ -meson mass m_ρ^2 and the effective continuum threshold s_0 .

The strategy will be just the same as used for the oscillator states:

- transfer the continuum contribution to the right hand side,
- write the original sum rule in the form

$$f_\rho^2 e^{-m_\rho^2/M^2} = \frac{3}{2} M^2 \left(1 - e^{-s_0/M^2}\right) + \text{“power corrections”}, \quad (158)$$

where, for brevity, we introduced the notation $f_\rho^2 = \frac{12\pi^2 m_\rho^2}{g_\rho^2}$,

- write the daughter sum rule in the form

$$f_\rho^2 m_\rho^2 e^{-m_\rho^2/M^2} = \frac{3}{2} M^4 \left(1 - \left(1 + \frac{s_0}{M^2}\right) e^{-s_0/M^2}\right) + \text{“power corrections”}, \quad (159)$$

- divide the second sum rule by the first to get the expression for m_ρ^2 as a function of s_0 and M^2 (for fixed values of the condensates),
- look for the value of the effective threshold s_0 that produces the most stable curve for m_ρ^2 (recall that with all power corrections taken into account, and the exact ansatz for the higher states, one should obtain an M^2 -independent result for m_ρ^2).

In this way one would obtain^b $s_0 \approx 1.5 \text{ GeV}^2$ and $m_\rho^2 \approx 0.58 \text{ GeV}^2$. Next step is to extract g_ρ^2 from the first sum rule, using these values of s_0 and m_ρ^2 . The value $s_0 \approx 1.5 \text{ GeV}^2$ provides the most stable curve for g_ρ^2 , corresponding to $g_\rho^2 \approx 2.17$. The result is rather sensitive to the m_ρ^2 -value. If one takes the experimental value $m_\rho^2 \approx 0.6 \text{ GeV}^2$, then the most stable curve still is that for $s_0 \approx 1.5 \text{ GeV}^2$, but the g_ρ^2 -value is a little higher $g_\rho^2 \approx 2.40$ and closer to the experimental one $g_\rho^2 \approx 2.36 \pm 0.18$. Anyway, it should be emphasized that all the g_ρ^2 -values extracted from the QCD sum rule differ from the experimental value by less than 10%, the claimed precision of the method.

6 QCD Sum Rules for Pion

6.1 Axial vector current

Vector currents are a special case, because the relevant correlators can be related to directly measurable cross sections. From purely theoretical point of view, other currents such as scalar, pseudoscalar, axial vector etc., are not worse, but experimental information in these cases is normally limited to the masses of the lowest states. Still, the axial vector current is in a better situation. First, its projection onto the pion state is just the pion decay constant f_π :

$$\langle 0 | \bar{d} \gamma_5 \gamma_\mu u | P \rangle = i f_\pi P_\mu. \quad (160)$$

Second, measuring the decays of the heavy lepton $\tau \rightarrow \nu_\tau + X$, one can measure the coupling of a hadronic state X to the axial current. Our goal now is to show how one can use the QCD sum rules to study the hadronic spectrum in the axial vector channel. Again, our presentation here closely follows the classic SVZ analysis¹.

Since the axial vector current $a_\mu(x) = \bar{d} \gamma_5 \gamma_\mu u$ is not conserved, the relevant

^bWith all the numbers explicitly given, the readers are encouraged to perform the fitting as an exercise.

correlator $\Pi_{\mu\nu}^{(ax)}(q)$ is a sum of two independent functions:

$$\Pi_{\mu\nu}^{(ax)}(q) = i \int e^{iqx} \langle 0 | T(a_\mu^+(x) a_\nu^-(0)) | 0 \rangle d^4x = -g_{\mu\nu} \Pi_1(Q^2) + q_\mu q_\nu \Pi_2(Q^2), \quad (161)$$

or, in terms of the transverse and longitudinal components

$$\Pi_{\mu\nu}^{(ax)}(q) = - \left\{ g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right\} \Pi_1(Q^2) - \frac{q_\mu q_\nu}{q^2} \{ \Pi_1(Q^2) + Q^2 \Pi_2(Q^2) \}. \quad (162)$$

6.2 Massless pion

The common belief is that the pion mass vanishes in the limit of exact chiral symmetry so that the pion is a Goldstone particle. It can be shown, that the operator product expansion in this case really requires the presence of a massless particle in the limit $m_q \rightarrow 0$. In this limit the imaginary part of the longitudinal function $\Pi_1(Q^2) + Q^2 \Pi_2(Q^2) \equiv \Pi_{||}$ vanishes, and $\Pi_{||}$ is a polynomial in Q^2 . Now, let us switch on a small quark mass and keep terms linear in this mass. The statement is that the function $\Pi_1(Q^2) + Q^2 \Pi_2(Q^2)$ is exactly calculable in this approximation:

$$\Pi_1(Q^2) + Q^2 \Pi_2(Q^2) = \frac{(m_u + m_d) \langle \bar{u}u + \bar{d}d \rangle}{Q^2} + O(m_q^2). \quad (163)$$

There are no terms linear in m_q of higher orders in $1/Q^2$. Let us show that this equation really implies the existence of a nearly massless pion. First we should express the Π -functions in terms of the hadronic contributions, using the dispersion relations:

$$\Pi_1(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{\text{Im } \Pi_1}{s + Q^2} ds + \text{"subtr."} \quad (164)$$

$$\Pi_2(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{\text{Im } \Pi_2}{s + Q^2} ds + \text{"subtr."} \quad (165)$$

$$\Pi_1(Q^2) + Q^2 \Pi_2(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{\text{Im } \Pi_1 - s \text{Im } \Pi_2}{s + Q^2} ds + \text{"subtr."} \quad (166)$$

where $\text{Im } \Pi_1$ contains contributions only from pseudovector (spin-1) states and their contribution should be transverse, i.e., proportional to $(g_{\mu\nu} - q_\mu q_\nu / q^2)$:

$$\text{Im } \Pi_1 = \sum_A \pi f_A^2 \delta(s - M^2), \quad (167)$$

while $\text{Im } \Pi_2$ contains contributions both from pseudovector (spin-1) states and from the pseudoscalar (spin-0) ones and their contribution into $\Pi_{\mu\nu}^{(ax)}(q)$ should be longitudinal *i.e.*, proportional to $q_\mu q_\nu$:

$$s\text{Im}\Pi_2 = \sum_A \pi f_A^2 \delta(s - M_A^2) + \sum_P \pi f_P^2 m_P^2 \delta(s - M_P^2). \quad (168)$$

Hence, only pseudoscalar states contribute to the longitudinal component of $\Pi_{\mu\nu}^{(ax)}$. For small quark masses we have

$$\begin{aligned} \frac{(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle}{Q^2} + O(m_q^2) &= - \sum_P \frac{f_P^2 m_P^2}{Q^2 + m_P^2} = \\ &= - \sum_P \left(\frac{f_P^2 m_P^2}{Q^2} - \frac{f_P^2 m_P^4}{Q^4} + \dots \right). \end{aligned} \quad (169)$$

All $O(Q^{-4})$ terms should be $O(m_q^2)$. This means that

$$f_P^2 m_P^4 = O(m_q^2) \quad (170)$$

for all states. Hence, the particles with masses that remain constant as $m_q \rightarrow 0$, decouple in the chiral limit:

$$m_P^2 = O(m_q^0), \quad (171)$$

$$f_P^2 = O(m_q^2). \quad (172)$$

The last relation also means that they do not contribute to the term linear in m_q . The only way to get this term is to assume that there exists a state for which the $O(m_q)$ -term is brought by the m_P^2 -factor:

$$m_P^2 = O(m_q^1), \quad (173)$$

$$f_P = O(m_q^0). \quad (174)$$

Such a state is naturally identified with the pion. In addition, we get the well-known relation

$$f_\pi^2 m_\pi^2 = -(m_u + m_d)\langle \bar{u}u + \bar{d}d \rangle \quad (175)$$

demonstrating explicitly the vanishing of the pion mass in the chiral limit.

To derive the starting statement, one should consider $q_\mu q_\nu \Pi_{\mu\nu}^{(ax)}(q)$. By equations of motion it is related to the correlator of the pseudoscalar densities:

$$\begin{aligned} q_\mu q_\nu \Pi_{\mu\nu}^{(ax)}(q) &= Q^2(\Pi_1(Q^2) + Q^2 \Pi_2(Q^2)) = \\ &= -i \int e^{iqx} \langle 0 | T \left((\bar{u}(x) \gamma_5 d(x)) (\bar{d}(0) \gamma_5 u(0)) \right) | 0 \rangle d^4x + \text{const.} \end{aligned} \quad (176)$$

The constant on the right-hand side is due to contact terms which normally arise if one differentiates a T -product. This constant corresponds to the $1/Q^2$ -term in the combination $\Pi_1(Q^2) + Q^2\Pi_2(Q^2)$. Direct calculation of this term gives the result stated in the beginning of this subsection.

Thus, the operator product expansion implies that $m_\pi^2 \rightarrow 0$ in the $m_q \rightarrow 0$ limit, and, hence, the pion mass should not be fitted from the sum rule. Within our accuracy one can set $m_\pi^2 = 0$ from the start.

6.3 QCD sum rule for the axial channel

The operator product expansion for the correlator of two axial currents is completely analogous to that of two vector currents, the only difference being the presence of the extra γ_5 -matrices in the vertices corresponding to the currents. For massless quarks, however, this produces no changes both in perturbative and $O(\langle GG \rangle)$ contributions. However, the four-quark terms shown in Fig.7e,f change sign, because one of the γ -matrix type terms is substituted by the condensate factor $\langle \bar{q}q \rangle$ which has the structure of $\mathbf{1}$ with respect to the Dirac indices.

After the Borel transformation the sum rule looks as follows:

$$\begin{aligned} \int_0^\infty e^{-s/M^2} \text{Im } \Pi_2(s) ds = & \frac{M^2}{4\pi} \left[1 + \frac{\alpha_s(M)}{\pi} + \frac{\pi^2}{3M^4} \langle \frac{\alpha_s}{\pi} GG \rangle + \right. \\ & + \frac{4\pi^3}{M^6} \langle \alpha_s (\bar{u}\gamma_\alpha\gamma_5\tau^a d \bar{d}\gamma_\alpha\gamma_5\tau^a u) \rangle + \\ & \left. + \frac{4\pi^3}{9M^6} \langle \alpha_s (\bar{u}\gamma_\alpha\tau^a u + \bar{d}\gamma_\alpha\tau^a d) \sum_{q=u,d,s} \langle 0 | \bar{q}\gamma_\mu\tau^a q | 0 \rangle \right]. \quad (177) \end{aligned}$$

In the axial channel we have two states that are the lowest among other states with the same quantum numbers: a pseudoscalar (the pion) and an axial vector (A_1 -meson) ones. Thus, one can try the ansatz with two low-lying states (“pion+ A_1 +continuum”):

$$\text{Im } \Pi_2(s) = \pi f_\pi^2 \delta(s) + \pi f_{A_1}^2 \delta(s - m_{A_1}^2) + \frac{1}{4\pi} \left(1 + \frac{\alpha_s}{\pi} \right) \theta(s > s_0^{(A_1)}). \quad (178)$$

One can also treat the A_1 as a part of the continuum. This, more crude approximation, corresponds to the simplified ansatz

$$\text{Im } \Pi_2(s) = \pi f_\pi^2 \delta(s) + \frac{1}{4\pi} \left(1 + \frac{\alpha_s}{\pi} \right) \theta(s > s_0^{(\pi)}). \quad (179)$$

The constants f_π, f_{A_1} are defined as follows:

$$\langle 0 | \bar{d}\gamma_5\gamma_\mu u | \pi \rangle = i f_\pi P_\mu, \quad \langle 0 | \bar{d}\gamma_5\gamma_\mu u | A_1 \rangle = i f_{A_1} \epsilon_\mu, \quad (180)$$

where ϵ_μ is the A_1 polarization vector.

6.4 Fitting the sum rule

Substituting the numerical values of the condensates into the basic sum rule gives

$$\int_0^\infty e^{-s/M^2} \text{Im } \Pi_2(s) ds = \frac{M^2}{4\pi} \left[1 + \frac{\alpha_s(M)}{\pi} + 0.1 \left(\frac{0.6 \text{GeV}^2}{M^2} \right)^2 + 0.22 \left(\frac{0.6 \text{GeV}^2}{M^2} \right)^3 \right]. \quad (181)$$

The “daughter” sum rule is

$$\int_0^\infty e^{-s/M^2} \text{Im } \Pi_2(s) s ds = \frac{M^2}{4\pi} \left[1 + \frac{\alpha_s(M)}{\pi} - 0.1 \left(\frac{0.6 \text{GeV}^2}{M^2} \right)^2 - 0.44 \left(\frac{0.6 \text{GeV}^2}{M^2} \right)^3 \right]. \quad (182)$$

Note, that in the $m_\pi^2 = 0$ approximation the pion does not contribute to the daughter sum rule, and the spectrum structure for $s \text{Im } \Pi_2(s) = \text{Im } \Pi_1(s)$ is more like in the vector channel: first non-zero mass resonance + continuum. This is reflected in the structure of the relevant sum rule: power corrections are negative. They are considerably larger than those in the ρ -channel, and one should expect that the A_1 -mass (squared) is much larger than m_ρ^2 .

The fitting procedure^c is most straightforward in the case of the simplified ansatz. One should transfer the continuum contribution to the right hand side to get f_π^2 as a function of $s_0^{(\pi)}$ and the Borel parameter M^2 :

$$4\pi^2 f_\pi^2 = M^2 \left(1 - e^{-s_0^{(\pi)}/M^2} \right) \left[1 + \frac{\alpha_s(M)}{\pi} \right] + \frac{\pi^2}{3M^2} \langle \frac{\alpha_s}{\pi} GG \rangle + \frac{704\pi^3}{M^4} \alpha_s \langle \bar{q}q \rangle^2. \quad (183)$$

The most stable curve corresponds to $s_0 \approx 0.7 \text{ GeV}^2$ and $4\pi^2 f_\pi^2 \approx 0.65 \text{ GeV}^2$, the experimental value being $4\pi^2 f_\pi^2 = 0.67 \text{ GeV}^2$.

One can also try the ansatz in which A_1 is treated as a (narrow) resonance. The strategy is just the same as in the ρ -case. To this end, one should

- differentiate the daughter (second) sum rule with respect to $1/M^2$ to get the third sum rule,
- transfer the continuum contribution to the right hand side in all sum rules,

^cAgain, we encourage the readers to obtain the curves themselves.

- extract $m_{A_1}^2$ and $s_0^{(A_1)}$ from the ratio of the third sum rule to the second,
- using the values $m_{A_1}^2 \approx 1.6 \text{ GeV}^2$ and $s_0^{(A_1)} \approx 2.4 \text{ GeV}^2$ obtained from the previous fitting, find $f_{A_1}^2$ from the second sum rule,
- using the value $4\pi^2 f_{A_1}^2 \approx 1.95 \text{ GeV}^2$ obtained from the previous fitting, find f_π^2 from the first sum rule. The value $f_\pi^2 \approx 0.69 \text{ GeV}^2$ obtained in this way is very close to the experimental one $f_\pi^2|^{exp} = 0.67 \text{ GeV}^2$. The A_1 mass value $m_{A_1}^2 \approx 1.6 \text{ GeV}^2$ extracted from the QCD sum rule also is in a very good agreement with experiment.

7 QCD Sum Rules for Pion Form Factor

7.1 Three-point function

To analyze the pion form factor, we consider the three-point function, *i.e.*, the correlator of the three currents^{20,21}:

$$T_{\alpha\beta}^\mu(p_1, p_2) = i \int e^{-ip_1 x + ip_2 y} \langle \{j_\beta(y) J^\mu(0) j_\alpha^+(x)\} \rangle d^4 x d^4 y, \quad (184)$$

where

$$J^\mu = \frac{2}{3} \bar{u} \gamma^\mu u - \frac{1}{3} \bar{d} \gamma^\mu d \quad (185)$$

is the electromagnetic current and

$$j_\alpha = \bar{d} \gamma_5 \gamma_\alpha u \quad (186)$$

is the axial current used in the preceding section to study the static properties of the pion. The pion contribution

$$\langle 0 | j_\beta(y) | p_2 \rangle \langle p_2 | J^\mu(0) | p_1 \rangle \langle p_1 | j_\alpha^+(x) | 0 \rangle \quad (187)$$

into this three-point correlator can be obtained by inserting the pion state(s) between the currents. The form factor $F_\pi(q^2)$ appears in the middle matrix element:

$$\langle p_2 | J^\mu(0) | p_1 \rangle = (p_1^\mu + p_2^\mu) F_\pi(q^2), \quad (188)$$

while the other two are projections of the axial current onto the initial and final pion states

$$\langle 0 | j_\beta(0) | p_2 \rangle = i f_\pi (p_2)_\beta \quad (189)$$

$$\langle p_1 | j_\alpha^+(0) | 0 \rangle = i f_\pi (p_1)_\alpha. \quad (190)$$

The three-point function $T_{\alpha\beta}^\mu(p_1, p_2)$ is the sum of different structures each characterized by the relevant invariant amplitude $t_i(p_1^2, p_2^2, q^2)$. The first idea is

to study the pion form factor analyzing the invariant amplitude corresponding to the structure $(p_2)_\beta(p_1)_\alpha(p_1^\mu + p_2^\mu)$. However, there are other structures $((p_1)_\beta(p_2)_\alpha, (p_1)_\beta(p_1)_\alpha, (p_2)_\beta(p_2)_\alpha)$ that coincide with $(p_2)_\beta(p_1)_\alpha$ in the limit $q \rightarrow 0$. This complication disappears if all the basic structures are expanded in $P = (p_1 + p_2)/2$ and $q = p_2 - p_1$. Then the pion contribution is

$$(p_2)_\beta(p_1)_\alpha(p_1^\mu + p_2^\mu) = 2 \left(P_\beta + \frac{q_\beta}{2} \right) \left(P_\alpha - \frac{q_\alpha}{2} \right) P^\mu = 2P_\beta P_\alpha P^\mu + O(q),$$

and the “best” structure (not changing when q varies) is $P_\alpha P_\beta P^\mu$. The simplest way to extract the relevant amplitude (hereafter referred to as T) is to multiply $T_{\alpha\beta}^\mu(p_1, p_2)$ by $n_\mu n^\alpha n^\beta$, where n is a lightlike vector orthogonal to q . The property $n^2 = 0$ kills all the structures containing $g_{\alpha\beta}$ and other g -terms, while the property $(nq) = 0$ kills all the structures containing any q -factor. Thus, the amplitude we are going to analyze is

$$T(p_1^2, p_2^2, q^2) = \frac{n_\mu n^\alpha n^\beta}{2(nP)^3} T_{\alpha\beta}^\mu(p_1, p_2). \quad (191)$$

7.2 Dispersion representation

To extract information about the form factors of physical states, one should incorporate the double dispersion relation

$$T(p_1^2, p_2^2, q^2) = \frac{1}{\pi^2} \int_0^\infty ds_1 \int_0^\infty ds_2 \frac{\rho(s_1, s_2, q^2)}{(s_1 - p_1^2)(s_2 - p_2^2)} + \text{“subtractions”}. \quad (192)$$

The subtraction terms, in general, are polynomial in p_1^2 and/or p_2^2 . They disappear after one applies $B(p_1^2 \rightarrow M_1^2)B(p_2^2 \rightarrow M_2^2)$, the Borel transformation in both variables^d. The double Borel transform $\Phi = B_1 B_2 T$ is related to the double spectral density $\rho(s_1, s_2, q^2 = -Q^2)$ by

$$\Phi(M_1^2, M_2^2, q^2 = -Q^2) = \frac{1}{\pi^2} \int_0^\infty \frac{ds_1}{M_1^2} \int_0^\infty \frac{ds_2}{M_2^2} \exp\left(-\frac{s_1}{M_1^2} - \frac{s_2}{M_2^2}\right) \rho(s_1, s_2, Q^2). \quad (193)$$

7.3 Operator product expansion

In contrast to the two-point correlators $\Pi(Q^2)$ analyzed in the previous Section, the three-point amplitudes $T(p_1^2, p_2^2, q^2)$ have three arguments. To incorporate

^d In our specific case, the subtractions are exhausted by a constant, and a single differentiation with respect to any variable is sufficient to remove it.

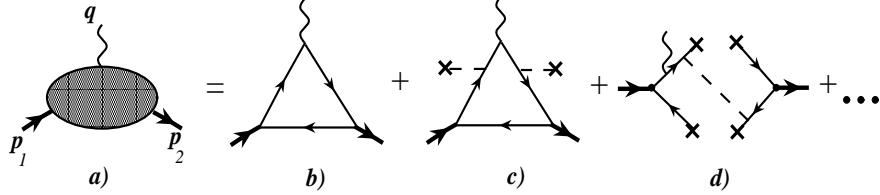


Figure 8: Structure of OPE in case of symmetric kinematics.

asymptotic freedom, one should take the invariants p_1^2, p_2^2 corresponding to the “hadronized” channels to be negative and sufficiently large, say, $|p_1^2|, |p_2^2| > 1 \text{ GeV}^2$. Depending on the value of the third invariant q^2 that works as an external parameter (it also should be negative, of course), there are three essentially different situations.

- The simplest is the symmetric case when Q^2 is of the same order of magnitude as $|p_1^2|, |p_2^2|$:

$$|p_1^2| \sim |p_2^2| \sim Q^2 \sim \mu^2.$$

The operator product expansion for T in this case has the following structure:

$$T(p_1^2, p_2^2, q^2) = \sum_i \frac{1}{(\mu^2)^{d_i}} C_i \left(\frac{\mu^2}{p_1^2}, \frac{\mu^2}{p_2^2}, \frac{\mu^2}{q^2} \right) \langle \mathcal{O}_i \rangle \quad (194)$$

where $C_i/(\mu^2)^{d_i}$ is a perturbatively calculable short-distance coefficient function, $\langle \mathcal{O}_i \rangle$ is the vacuum matrix element of a local operator, *i.e.*, some condensate term and d_i is the dimension of \mathcal{O}_i in mass units (see Fig.8).

- The case when Q^2 is small is more complicated:

$$|p_1^2| \sim |p_2^2| \sim \mu^2, \quad Q^2 \ll \mu^2.$$

The coefficient functions C_i calculated in the symmetric kinematics, might become singular in this limit because of $(\mu^2/Q^2)^n$ - or $\log(\mu^2/Q^2)$ -terms. The operator product expansion in such a situation has a modified form:

$$\begin{aligned} T(p_1^2, p_2^2, q^2) &= \sum_i \frac{1}{(\mu^2)^{d_i}} \tilde{C}_i \left(\frac{\mu^2}{p_1^2}, \frac{\mu^2}{p_2^2}, \frac{\mu^2}{q^2} \right) \langle \mathcal{O}_i \rangle \\ &+ \sum_j \frac{1}{(\mu^2)^{d_j}} E_k \left(\frac{\mu^2}{p_1^2}, \frac{\mu^2}{p_2^2} \right) \Pi_k(Q^2) \end{aligned} \quad (195)$$

where the coefficient functions \tilde{C}_i are regular in the $Q^2 \rightarrow 0$ limit, $\Pi_k(Q^2)$ is the correlator of the electromagnetic current J and a local operator \mathcal{O}_k

$$\Pi_k(Q^2) = i \int e^{-iqx} \langle \mathcal{O}_k(x) J(0) \rangle d^4x,$$

d_k is the dimension of \mathcal{O}_k in mass units. The coefficient functions E_k are (the Fourier transforms of) those for the operator product expansion of two axial currents

$$T(j j^+) = \sum_k E_k \mathcal{O}_k.$$

In the modified OPE for the three-point function, the contributions of the first type correspond to a situation when large momentum flow connects all three external vertices, while those of the second type describe the situation when large momentum flows only between the vertices x and y related to the axial currents. The momenta in the J -channel are of the order of Q , *i.e.*, small, and the correlator $\Pi_j(Q^2)$ is a nonperturbative object accumulating long-distance information.

- In the opposite limit of large Q^2

$$|p_1^2| \sim |p_2^2| \sim \mu^2, \quad Q^2 \gg \mu^2,$$

a simple expansion in powers of $1/|p_1^2|, 1/|p_2^2|$ (*i.e.*, in $1/\mu^2$) might be destabilized by large $(Q^2/\mu^2)^n$ -terms. Appearance of terms growing with Q^2 is a bit surprising, since form factors should decrease as $Q^2 \rightarrow \infty$. But the appearance of such terms is just the artifact of the $1/\mu^2$ expansion: even a very reasonable function like $1/(Q^2 + \mu^2)$, vanishing as $Q^2 \rightarrow \infty$, produces a divergent series when expanded in $1/\mu^2$. Thus, a possible way out in the large- Q^2 limit is a resummation of the parametrically enhanced contributions of $(Q^2/\mu^2)^n$ -type. In fact, it can be established, that such contributions result from the Taylor expansion of the original nonlocal combinations $\langle \varphi(0)\varphi(x) \rangle$ (nonlocal condensates), and the idea is to construct sum rules without expanding the nonlocal condensates, treating them as some phenomenological functions describing the propagation of quark and gluons in the physical vacuum.

7.4 Perturbative contribution

The structure of the power corrections in the operator product expansion, as we discussed above, depends on the particular Q^2 -region under study. However, the starting point of any expansion is the perturbative contribution, in our case corresponding to the triangle diagram plus radiative corrections to it.

Using Feynman parametrization, one can rather easily obtain the following representation²¹ for the double Borel transform of the lowest-order perturbative triangle diagram (see Fig.8a)

$$\Phi^{pert}(M_1^2, M_2^2, Q^2) = \frac{3}{2\pi^2(M_1^2 + M_2^2)} \int_0^1 x(1-x) \exp\left\{-\frac{xQ^2}{(1-x)(M_1^2 + M_2^2)}\right\} I(x, M_1^2, M_2^2, m_q^2, e_q) dx \quad (196)$$

where $I(x, M_1^2, M_2^2, m_q^2, e_q)$ is the factor containing the dependence on the quark masses

$$I(x, M_1^2, M_2^2, m_q^2, e_q) = e_u \exp\left\{-\left(\frac{m_u^2}{1-x} + \frac{m_d^2}{x}\right)\left(\frac{1}{M_1^2} + \frac{1}{M_2^2}\right)\right\} + \{u \leftrightarrow d\}. \quad (197)$$

In view of extreme smallness of the u - and d -quark masses, this factor can be safely set to unity. The x -variable may be interpreted as the fraction of the total momentum P carried by the passive quark (active is the quark interacting with the EM current). This integral representation is very convenient to study the behavior of $\Phi^{pert}(M_1^2, M_2^2, Q^2)$ for small and large Q^2 . In the low- Q^2 limit we get

$$\Phi^{pert}(M_1^2, M_2^2, Q^2) = \frac{1}{4\pi^2(M_1^2 + M_2^2)} \left\{ 1 - 2\frac{Q^2}{M_1^2 + M_2^2} - \sum_{N=0}^{\infty} \left(\frac{-Q^2}{M_1^2 + M_2^2}\right)^{N+2} \frac{N+3}{N!} \left(\log\left[\frac{Q^2}{M_1^2 + M_2^2}\right] + \frac{1}{N+3} - \psi(N+1)\right) \right\}. \quad (198)$$

The first two terms of the expansion can be obtained by simply expanding the exponential factor in Q^2 under the integral sign. However, the third term of such an expansion produces a logarithmically divergent integral

$$\cdots \int_0^1 \frac{1}{2!} \left(\frac{Q^2}{M_1^2 + M_2^2}\right)^2 \frac{x^3}{1-x} dx,$$

and higher terms have power divergences when integrated in the region $x \rightarrow 1$. This region corresponds to a situation when the whole large momentum P is carried by the passive quark, and the active quark carries small momentum proportional to q . This is precisely the situation producing additional terms in the operator product expansion at small Q^2 . The logarithmic factor $\log(Q^2)$, singular when $Q^2 \rightarrow 0$, just signalizes that there are long-distance

effects (quarks propagate over distances $\sim 1/Q$) limiting the applicability of perturbation theory.

In the large- Q^2 limit one should expand $\Phi^{pert}(M_1^2, M_2^2, Q^2)$ in $1/Q^2$ rather than in Q^2 :

$$\begin{aligned} \Phi^{pert}(M_1^2, M_2^2, Q^2) = & \frac{3}{2\pi^2} \frac{M_1^2 + M_2^2}{(Q^2)^2} \left\{ 1 - 8 \frac{M_1^2 + M_2^2}{Q^2} + \right. \\ & \left. + \frac{1}{6} \sum_{N=2}^{\infty} (-)^N (N+1)(N+3)! \left(\frac{M_1^2 + M_2^2}{Q^2} \right)^N \right\}. \end{aligned} \quad (199)$$

Thus, the perturbative contribution vanishes like $1/Q^4$ as $Q^2 \rightarrow \infty$. Combining the spectral representation for $\Phi(M_1^2, M_2^2, Q^2)$ and the explicit expression for it, we get the equation for the perturbative spectral density $\rho^{pert}(s_1, s_2, Q^2)$

$$\begin{aligned} & \frac{1}{\pi^2} \int_0^\infty \frac{ds_1}{M_1^2} \int_0^\infty \frac{ds_2}{M_2^2} \exp\left(-\frac{s_1}{M_1^2} - \frac{s_2}{M_2^2}\right) \rho^{pert}(s_1, s_2, Q^2) = \\ & = \frac{3}{2\pi^2(M_1^2 + M_2^2)} \int_0^1 x(1-x) \exp\left(-\frac{xQ^2}{(1-x)(M_1^2 + M_2^2)}\right) dx. \end{aligned} \quad (200)$$

Solving it (this amounts to a double inverse Laplace transformation) gives the perturbative spectral density:

$$\rho^{pert}(s_1, s_2, Q^2) = \frac{3}{2} \left[\frac{(Q^2)^2}{2!} + \frac{(Q^2)^3}{3!} \right] \frac{1}{\sqrt{(s_1 + s_2 + Q^2)^2 - 4s_1 s_2}}. \quad (201)$$

In the low- Q^2 limit the perturbative spectral density can be expanded in Q^2 :

$$\begin{aligned} \rho^{pert}(s_1, s_2, Q^2)|_{Q^2 \rightarrow 0} = & \frac{1}{4} \delta(s_1 - s_2) \theta(s_1) \theta(s_2) + \\ & + \frac{Q^2}{4} (s_1 + s_2) \delta''(s_1 - s_2) \theta(s_1) \theta(s_2) + \dots \end{aligned} \quad (202)$$

For large Q^2 one can expand it in $1/Q^2$:

$$\rho^{pert}(s_1, s_2, Q^2)|_{Q^2 \rightarrow \infty} = \frac{3}{2} \theta(s_1) \theta(s_2) \left\{ \frac{s_1 + s_2}{Q^4} - 4 \frac{s_1^2 + s_2^2 + 4s_1 s_2}{Q^6} + \dots \right\}. \quad (203)$$

Thus, for small Q^2 , the perturbative spectral density is concentrated on the line $s_1 = s_2$: there are no transitions between states with different masses. Along this line, the spectral density does not vary: it has no dependence on $s_1 + s_2$. In the opposite limit, when $Q^2 \rightarrow \infty$, the situation is, in a sense,

reversed: the density is nonzero in the whole region $s_1 > 0, s_2 > 0$, there is no dependence on $(s_1 - s_2)$, and the density grows like $s_1 + s_2$. The last observation means that higher states are relatively more important for large Q^2 than for the small ones.

7.5 Sum rule for the pion form factor at moderate Q^2

The physical spectral density $\rho(s_1, s_2, Q^2)$, of course, differs from its perturbative analog. It has a rather complicated structure on the s_1, s_2 -plane. In particular, $\rho(s_1, s_2, Q^2)$ contains the term corresponding to the pion form factor

$$\rho_{\pi\pi}(s_1, s_2, Q^2) = \pi^2 f_\pi^2 F_\pi(Q^2) \delta(s_1 - m_\pi^2) \delta(s_2 - m_\pi^2). \quad (204)$$

In addition, it contains the contributions corresponding to transitions between the pion and higher resonances, and also the terms related to elastic and transition form factors of the higher resonances. Constructing the two-dimensional (s_1, s_2) analog of the simplest ansatz “lowest state plus continuum” we can treat all the contributions, except for the $\rho_{\pi\pi}$, as “continuum”, *i.e.*, assume that $\rho(s_1, s_2, Q^2) = \rho^{pert}(s_1, s_2, Q^2)$ outside the square $(0, s_0) \times (0, s_0)$:

$$\rho(s_1, s_2, Q^2) = \rho_{\pi\pi}(s_1, s_2, Q^2) + (1 - \theta(s_1 < s_0)\theta(s_2 < s_0)) \rho^{pert}(s_1, s_2, Q^2). \quad (205)$$

Now, transferring the continuum contribution to the right-hand side of the sum rule and using the operator product expansion in the symmetric kinematics ($Q^2 \sim M_1^2, M_2^2$), we obtain the QCD sum rule for the pion form factor:

$$f_\pi^2 F_\pi(Q^2) = \frac{1}{\pi^2} \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \exp\left(-\frac{s_1 + s_2}{M^2}\right) \rho^{pert}(s_1, s_2, Q^2) + \frac{\alpha_s \langle GG \rangle}{12\pi M^2} + \frac{16 \pi \alpha_s \langle \bar{q}q \rangle^2}{81 M^4} \left(13 + 2 \frac{Q^2}{M^2}\right). \quad (206)$$

To treat initial and final states on equal footing and to simplify the analysis, we took $M_1^2 = M_2^2$. Unfortunately, the double integral in this sum rule cannot be calculated explicitly for arbitrary Q^2 . However, it is instructive to consider the formal limit $Q^2 = 0$:

$$f_\pi^2 F_\pi(0) = \frac{1}{4\pi^2} \frac{M^2}{2} \left(1 - e^{-2s_0/M^2}\right) + \frac{\alpha_s \langle GG \rangle}{12\pi M^2} + \frac{208 \pi \alpha_s \langle \bar{q}q \rangle^2}{81 M^4}. \quad (207)$$

This should be compared to the f_π^2 sum rule:

$$f_\pi^2 = \frac{m^2}{4\pi^2} \left(1 - e^{-s_0/m^2}\right) + \frac{\alpha_s \langle GG \rangle}{12\pi m^2} + \frac{176 \pi \alpha_s \langle \bar{q}q \rangle^2}{81 m^4}. \quad (208)$$

The condensate terms look almost identical, but comparing the perturbative terms we observe that M^2 , the Borel parameter of the form factor sum rule, should be larger by factor 2 than m^2 , the Borel parameter of the f_π^2 sum rule: $M^2 = 2m^2$. After this change, the $\langle GG \rangle$ -term in the form factor sum rule is multiplied by $\frac{1}{2}$ and the $\langle \bar{q}q \rangle^2$ -term is multiplied by $\frac{1}{4}$

$$f_\pi^2 F_\pi(0) = \frac{m^2}{4\pi^2} \left(1 - e^{-s_0/m^2}\right) + \frac{\alpha_s \langle GG \rangle}{24\pi m^2} + \frac{52}{81} \frac{\pi \alpha_s \langle \bar{q}q \rangle^2}{m^4}. \quad (209)$$

Thus, we obtained a sum rule, which looks like that for f_π^2 , but with the gluonic condensate term smaller by factor $2 \approx (1.4)^2$, and the quark condensate term smaller by factor $\sim 3.5 \approx (1.5)^3$. Since it is the values of the condensates that (after the fitting procedure) determine all the hadronic scales, we conclude that all the hadronic parameters having the dimension of $(mass)^2$, *i.e.*, s_0 and the combination $f_\pi^2 F_\pi(0)$, extracted from this sum rule are by a factor $1.4 - 1.5$ smaller than s_0^π and f_π^2 , respectively^e. To get s_0 closer to s_0^π , one should restore the matching between the perturbative and the condensate terms, *i.e.*, decrease the perturbative term by a factor 2 to 4. This allows us to make an educated guess that QCD sum rule for the pion form factor will most effectively work in the intermediate region where the form factor varies between 0.5 and 0.25, *i.e.*, for Q^2 between 0.5 GeV^2 and 2 GeV^2 .

The readers can check these guesses by explicit fitting procedure. To this end, one should plot the combination $f_\pi^2 F_\pi(Q^2)$ as a function of the Borel parameter M^2 for different values of the effective threshold s_0 . As a “true” value of s_0 one can take the value for which the curve is constant as $M^2 \rightarrow \infty$. One would observe that the value of s_0 obtained in this way slowly grows with Q^2 , from 0.6 GeV^2 for $Q^2 = 0.5 \text{ GeV}^2$ to 1.0 GeV^2 for $Q^2 = 3 \text{ GeV}^2$. The growth of s_0 reflects the fact that, when the Q^2 value increases, the condensate contributions remain constant (with the quark term even slowly growing) whereas the perturbative term decreases. The constancy of the condensate contributions is an artifact of our approximation. If one includes operators of higher dimensions in the OPE, there appear terms diminishing the total condensate contribution for large Q^2 . These terms, however, have the structure $(Q^2/M^2)^n$, and one should resum them to get a reasonable result. However, if one restricts the analysis to the region $0.5 \text{ GeV}^2 < Q^2 < 3 \text{ GeV}^2$, the predictions of our sum rule agree with existing pion form factor data within 10% – 20% accuracy.

^eOne should not worry about the fact that a straightforward extrapolation of the form factor sum rule to the point $Q^2 = 0$, gives $F_\pi(0) \approx 0.7$. This sum rule is valid only for sufficiently high Q^2 , and, as we emphasized, there appear additional terms in the OPE which will increase the condensate contributions to make them equal to those in the f_π^2 sum rule and recover the $F_\pi(0) = 1$ normalization condition.

7.6 Local quark-hadron duality for the pion form factor

The approach described above allows to obtain the form factor at different values of Q^2 by independent fitting procedure, point by point. But it is tempting to see the Q^2 -curve as a whole. This can be done by looking at the $F_\pi(Q^2)$ -curves at fixed s_0 for different values of the Borel parameter M^2 . For $M^2 > 1.5 \text{ GeV}^2$, all the curves are close to each other, approaching the limiting (s_0 -dependent) form as $M^2 \rightarrow \infty$. The nature of this convergence is very simple: the condensate terms disappear from the sum rule in the $M^2 \rightarrow \infty$ limit, and the limiting curve corresponds to the local duality relation:

$$f_\pi^2 F_\pi(Q^2) = \frac{1}{\pi^2} \int_0^{s_0} ds_1 \int_0^{s_0} \rho^{pert}(s_1, s_2, Q^2) ds_2. \quad (210)$$

Note, that the function $\rho^{pert}(s_1, s_2, Q^2)$ describes the transition from a free-quark ($\bar{q}q$) state with invariant mass s_1 to a similar state with mass s_2 . The local duality relation states, essentially, that one can calculate the pion form factor by averaging the form factors of such transitions over the appropriate duality interval s_0 . In the two-point case, the local duality produces the relation between the pion decay constant f_π^2 and the duality interval s_0

$$s_0 = 4\pi^2 f_\pi^2 \approx 0.7 \text{ GeV}^2. \quad (211)$$

We recall that the duality interval is the effective threshold for production of higher resonances, and is very natural that it is just in the middle between $m_\pi^2 \approx 0$ and $m_{A_1}^2 \approx 1.6 \text{ GeV}^2$. Using the explicit form for $\rho^{pert}(s_1, s_2, Q^2)$, we obtain²¹

$$F_\pi(Q^2)|^{local\ duality} = 1 - \frac{1 + 6s_0/Q^2}{(1 + 4s_0/Q^2)^{3/2}}. \quad (212)$$

This formula gives a correct value for the form factor at $Q^2 = 0$, but the predicted slope is wrong:

$$F_\pi(Q^2)|^{local\ duality} = 1 - \frac{3}{4} \frac{Q}{\sqrt{s_0}} + \dots \quad (213)$$

This is just the same situation we encountered applying the local duality to the form factor of the lowest oscillator state. The reason is that there is a serious mismatch between perturbative and physical spectral densities: the perturbative density is concentrated in the narrow stripe $|s_1 - s_2| < Q$ near the $s_1 = s_2$ line, while physical density, in addition to the elastic form factors (points at the $s_1 = s_2$ line), contains also transition form factors (line $s_1 = 0, s_2 > s^{(3\pi\ threshold)}$, etc.) corresponding to the regions very far from the $s_1 =$

s_2 line. In other words, our model for the hadronic spectrum is too rough at small nonzero Q^2 . The perturbative density in this region can be approximated by

$$\rho^{pert}(s_1, s_2, Q^2) \sim \frac{1}{Q} \theta(|s_1 - s_2| < Q). \quad (214)$$

Integrating it over any line $s_1 + s_2 = \text{const.}$, one obtains a result analytic in Q^2 . In our model, however, there are small triangles not included into the integration region. Their area is $\sim Q^2$, and the density is $\sim 1/Q$, so the resulting contribution (to be subtracted from 1) is $O(Q) = O(\sqrt{Q^2})$. To avoid this, one can try a “triangle” model for the higher states: “pion + (continuum outside the triangle formed by the line $(s_1 + s_2 = S_0)$ ”. The local duality relation in this case gives²¹

$$F_\pi(Q^2)^{triangle\ l.d.} = \frac{S_0}{8\pi^2 f_\pi^2 (1 + Q^2/2S_0)^2}. \quad (215)$$

To get the correct normalization $F_\pi(0) = 1$ one should take $S_0 = 8\pi^2 f_\pi^2 = 2s_0$. However, the slope in this case is too small $F_\pi^{tr.l.d.}(Q^2)|_{S_0=2s_0} = 1 - Q^2/2s_0$, by factor 2 smaller than the experimental one, and the curve goes well above both the experimental points and the curve corresponding to the “squared” form of the local duality. To match with the latter, one should take $S_0 = \sqrt{2}s_0$. In this case the area of the local duality region is the same. Outside the small- Q^2 region, there is good agreement between the two local duality curves and the data approximated by the fit $F_\pi^{fit}(Q^2) \approx 1/(1 + Q^2/0.47)$.

7.7 One-gluon exchange contribution

The contribution into the pion form factor we calculated using the lowest order $O(\alpha_s^0)$ term in the perturbative spectral density corresponds to the so-called “soft diagram”. Since this contribution decreases as $1/Q^4$ for large Q^2 , it is normally ignored in the perturbative QCD analysis concentrated on the study of the “hard gluon exchange diagram” that has the $1/Q^2$ asymptotic behavior dictated by the quark counting rules.

However, the soft contribution as given, e.g., by the local quark-hadron duality, is very close to the experimental data leaving not much place for any other contribution. To estimate the contribution due to the “hard gluon” exchange diagrams, one should include the $O(\alpha_s(Q^2))$ -terms in the perturbative spectral density. To simplify the analysis, one can use the local duality approximation. Still, the two-loop calculation is rather complicated, but the results,

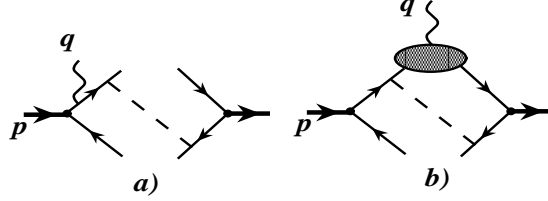


Figure 9: Gauge-invariant set of $\langle \bar{q}q \rangle^2$ diagrams.

with a rather high accuracy, can be approximated by a simple interpolation

$$\frac{1}{\pi^2 f_\pi^2} \int_0^{s_0} ds_1 \int_0^{s_0} ds_2 \Delta \rho^{pert}(s_1, s_2, q^2) \approx \frac{\alpha_s}{\pi} \frac{1}{1 + Q^2/2s_0} \quad (216)$$

between the $Q^2 = 0$ value (related by the Ward identity to the $O(\alpha_s)$ term of the 2-point correlator) and the asymptotic behavior $F_\pi^{as}(Q^2) = 8\pi\alpha_s(Q^2)f_\pi^2/Q^2$.

7.8 Pion form factor at small Q^2

As we discussed earlier, if the QCD sum rule derived in the moderate- Q^2 region is simple-mindedly extrapolated to the point $Q^2 = 0$, one obtains the expression

$$f_\pi^2 F_\pi(0) \stackrel{?}{=} \frac{m^2}{4\pi^2} \left(1 - e^{-s_0/m^2}\right) + \frac{\alpha_s \langle GG \rangle}{24\pi m^2} + \frac{52}{81} \frac{\pi \alpha_s \langle \bar{q}q \rangle^2}{m^4} \quad (217)$$

that differs from the sum rule for f_π^2

$$f_\pi^2 = \frac{m^2}{4\pi^2} \left(1 - e^{-s_0/m^2}\right) + \frac{\alpha_s \langle GG \rangle}{12\pi m^2} + \frac{176}{81} \frac{\pi \alpha_s \langle \bar{q}q \rangle^2}{m^4}. \quad (218)$$

The condensate terms have the coefficients essentially smaller than it is necessary to enjoy the simple Ward identity prediction $F_\pi(Q^2) = 1$. The latter requires that the QCD sum rule for $f_\pi^2 F_\pi(0)$ should coincide with that for f_π^2 . The Ward identity, being just the statement that the total electric charge of the pion is equal to that of the electron, should not be violated. Hence there should exist the additional condensate contributions “visible” at $Q^2 = 0$ (to satisfy the Ward identity) and dying rapidly for $Q^2 > m_\rho^2$ (to reproduce the intermediate- Q^2 sum rule).

To get a feeling about how these terms might look like, consider the $\alpha_s \langle \bar{q}q \rangle^2$ diagrams. To enjoy the consequences of the Ward identity, one should have a

gauge invariant set of diagrams. This can be achieved by inserting the photon vertex into all possible lines, including also the “external” lines corresponding to quarks going into vacuum. However, in perturbation theory, this brings in terms $\sim 1/Q^2$ singular in the $Q^2 \rightarrow 0$ limit. Such a “coefficient function” cannot be calculated perturbatively. Thus, one should separate the large- and small-momentum parts of the diagram

$$T(p^2, q^2) = C(p^2) \otimes \Pi(q^2) \quad (219)$$

and treat the Q^2 -dependent part as a correlator of the original electromagnetic current J and some local operator constructed from the quark fields going “into vacuum” (see Fig.9b):

$$\Pi(q^2) \sim \int e^{iqx} \langle T(J_\mu(0) \mathcal{O}(x)) \rangle d^4x. \quad (220)$$

The question now is whether it is possible to calculate contributions like the correlators $\Pi(q^2)$ at small q ? There are essentially two types of these correlators depending on the type of the operator \mathcal{O} . First, one should take into account that under the T -product sign, one cannot throw out the operators apparently vanishing due to the equations of motion, *e.g.*, those containing \not{D} acting on the quark field, since there appear contact terms. In this case $\Pi(q^2)$ is a constant proportional to the quark condensate $\langle \bar{q}q \rangle$. In all other cases one can (approximately) calculate $\Pi(q^2)$ using the following QCD sum rule strategy:

- Calculate $\Pi(q^2)$ for large Q^2 using the operator product expansion. This normally gives

$$\Pi(q^2 = -Q^2) \sim \frac{\langle \bar{q}q \rangle}{Q^2} + k \frac{\langle \bar{q}D^2q \rangle}{Q^4} + \dots$$

- Use the dispersion relation

$$\Pi(Q^2) = \frac{1}{\pi} \int_0^\infty \frac{\rho(s) ds}{s + Q^2}.$$

- Assume an appropriate ansatz for the spectral density:

$$\rho(s) = \pi A \delta(s - m_\rho^2) + \pi B \delta(s - m_R^2).$$

Since the perturbative density is zero in this case, the higher states (lying above the ρ -meson) are approximated by an effective resonance R .

- Requiring the best agreement between the two representations for $\Pi(Q^2)$,

extract the values of the constants A, B and m_R^2 .

- The result is

$$\Pi(Q^2) = \frac{A}{Q^2 + m_\rho^2} + \frac{B}{Q^2 + m_R^2},$$

and one can use it for small Q^2 .

In this way one can obtain the QCD sum rule for the pion form factor valid in the low- Q^2 region²²:

$$\begin{aligned} f_\pi^2 F_\pi(Q^2) = & \text{“perturbative part”} + \frac{\alpha_s \langle GG \rangle}{12\pi m^2} \left[\frac{1}{2} + \frac{m_\rho^2}{2(m_\rho^2 + Q^2)} \right] + \\ & + \frac{16 \pi \alpha_s \langle \bar{q}q \rangle^2}{81 m^4} \left[4 + 6 \left(\frac{1.6 m_\rho^2}{m_\rho^2 + Q^2} - \frac{0.6 m_R^2}{m_R^2 + Q^2} \right) + \frac{m_\rho^2}{m_\rho^2 + Q^2} + \frac{Q^2}{m^2} \right]. \end{aligned} \quad (221)$$

This sum rule possesses all the required properties:

- For $Q^2 = 0$ its left hand side exactly coincides with that of the f_π^2 sum rule.
- additional terms die out when Q^2 gets large.
- The curve for the pion form factor calculated with the help of this sum rule in the region of small Q^2 , at the matching point $Q^2 = m_\rho^2$, agrees within 10% with the curve obtained from the moderate- Q^2 sum rule.
- the pion charge radius, extracted from the low- Q^2 sum rule,

$$\langle r_\pi^2 \rangle^{1/2}|_{SR} = 0.66 \pm 0.03 \text{ fm}$$

is in good agreement with the experimental value

$$\langle r_\pi^2 \rangle^{1/2}|_{exp.} = 0.636 \pm 0.036 \text{ fm}.$$

8 Conclusions

Thus, the QCD sum rules give an accurate description of the pion form factor in the whole region where the data exist. They allow one to trace its behavior from the normalization point $Q^2 = 0$, through the low- Q^2 region, to the region of moderately large Q^2 . Everywhere the QCD sum rule results are in agreement with experiment within the accuracy of the method. The QCD sum rules unambiguously demonstrate that, in the experimentally accessible region, the form factor is dominated by the soft contribution, which is not calculable within a purely perturbative approach. They show that the one-gluon-exchange mechanism, though dominant in the asymptotic limit, is of minor importance in the region $Q^2 < 4 \text{ GeV}^2$ and, maybe, till even larger values of Q^2 .

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